Water Resources Systems
Modelling Techniques and Analysis
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Preface

A curriculum in water resources in teaching institutions across the country invariably contains at least one course on Water Resources Systems, which essentially deals with modelling techniques for optimum utilization of available water resources. Excellent books are available today, which specialize exclusively on individual topics of the course curriculum, such as linear programming, dynamic programming, and stochastic optimization with applications. However, a single book that caters to the needs of students entering the subject area, emphasizing the basics of systems techniques in water resources with illustrative examples, and potential applications to real systems, is preferable for classroom teaching. Also, most of the books available in the market are out of the affordable range of the students. These considerations motivated us in our attempt to bring out the book in this form.

The contents of the book are organized in three parts—Part One: Basics of Systems Techniques; Part Two: Model Development; and Part Three: Applications.

**Part One** presents the basic techniques necessary in water resources systems modelling. Chapter One gives a brief account of the concept of systems and systems analysis. The core part of optimization techniques is presented in Chapter Two. Basics of optimization using calculus, Linear Programming (LP) and Dynamic Programming (DP), and a brief account of simulation are covered in it. Chapter Three details the basic concepts of engineering economics. Microeconomics, price theory, demand curves, aggregation of demand, conditions of optimality in a production process, benefit and cost considerations, and evaluation of engineering alternatives are covered in this chapter. A brief account of multiobjective analysis is presented in Chapter Four, illustrating the basics of weighting and constraint methods of generating noninferior solutions to a multiobjective optimization problem.

**Part Two** illustrates the basic techniques in formulating models for selected problems in reservoir systems with deterministic as well as stochastic inputs. In Chapter Five models for reservoir sizing and operation, and reservoir simulation for hydropower production are discussed in the deterministic domain. Randomness of inflow is dealt with in Chapter Six, illustrating chance constrained linear programming and stochastic dynamic programming applications in reservoir operation.
Part Three presents a number of specific applications in reservoir systems modelling including recently developed tools: ANN and Fuzzy sets. Chapter Seven presents linear programming applications to crop yield optimization, reliability capacity relationships, multiobjective analysis of multireservoir system, short term reservoir operation, and reservoir operation for hydropower optimization. Dynamic programming is applied to optimal crop water allocation, derivation of steady state reservoir operating policy and real time operation in Chapter Eight. Chapter Nine introduces the basics of recent modelling tools—ANNs and Fuzzy sets. Applications illustrated include inflow forecasting using ANN, Fuzzy rule-based reservoir operation, and Fuzzy LP for river water quality management and reservoir operation. In the applications discussed in Part Three, model formulations of examples and case studies are presented without questioning the assumptions made in them, some of which may need explanation beyond the scope of the book. The aim is essentially to introduce the reader to the articulations in model formulation in a given situation with appropriate assumptions.

We have extensively used the contents of the first two parts of the book for teaching postgraduate students at Indian Institute of Science, Bangalore. Selected sections of the book may be used as teaching material for students at different levels: Part One to introduce the basics, Part Two to impart training to acquire skills in modelling, and Part Three to help articulation of model formulations to practical problems. Part One will in itself form the core contents of a 2-credit course in water resources systems in one semester. The course material may be supplemented by selected topics from Part Two, especially Chapter 5. A regular 3-credit course may include Parts One and Two. A term paper or a seminar may be included as a part of the syllabus, for which the contents of Part Three should aid students in selecting topics of individual interest. Graduate research students will find Part Three, in which a range of applications are illustrated, useful in selecting topics for further study or in generating additional skills for wider applications of systems modelling.

Over the past two and half decades, a large number of our students, endured our presence in the classroom and helped us learn with them. We thank them for helping us crystallize our thoughts over years in bringing out this book. We acknowledge here that we have been highly inspired by the classical book, “Water Resources Systems Planning and Analysis” by Loucks et al. (1981), which stimulated us to bring out this book in the present form.

Needless to say, in a maiden attempt of this kind, we might have inadvertently erred on some counts. We appeal to all readers, colleagues and students, to point out these, and to offer constructive suggestions for improving the book.

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PART 1

Basics of Systems Techniques

- Concept of System and Systems Analysis
- Systems Techniques in Water Resources
- Economic Considerations in Water Resources Systems
- Multiobjective Planning
This chapter deals with the definition of a system, factors governing a system, system analysis, typical problems associated with systems and a brief account of the techniques used to solve these problems. The concept of systems analysis is indeed very wide, but we shall confine our discussion in this section to a typical hydrological/water resources system.

1.1 DEFINITION OF A SYSTEM
The basic concept of a system is that it relates two or more things. Out of the several definitions of a system, the simplest one states that it is a device that accepts one or more inputs and generates one or more outputs. A comprehensive definition of a system is given by Dooge (Dooge, 1973) as “any structure, device, scheme, or procedure, real or abstract, that interrelates in a given time reference, an input, cause, or stimulus, of matter, energy, or information, and an output, effect or response, of information, energy or matter.” The input and output referred to in mechanics is synonymous with cause and effect in physics and philosophy, and stimulus and response in biological sciences. Typical examples of a system are a university with its various departments, a central government with its regional governments, a river basin with all its tributaries, and so on.

1.2 TYPES OF SYSTEMS
Some commonly understood types of systems are discussed as below.

Simple and Complex Systems
A simple system is one in which there is a direct relation between the input and the output of the system. A complex system is a combination of several sub-systems each of which is a simple system. Therefore, a complex system may be subdivided into a number of simple systems. Each subsystem has a distinct relation between input and output. For example, a river basin system is a
complex system comprising several subsystems, each corresponding to a tributary.

**Linear and Nonlinear Systems**

A linear system is one in which the output is a constant ratio of the input. In a linear system the output due to a combination of inputs is equal to the sum of the outputs from each of the inputs individually, i.e. the principle of superposition is valid. For example, a system (watershed) in which the input $x$ (rainfall) and the output $y$ (runoff) are related by $y = mx$, in which $m$ is a constant, is a linear system. The unit hydrograph in hydrology is a linear system (as the hydrograph ordinate of the direct runoff hydrograph is proportional to the rainfall excess). On the other hand, a system in which the input $x$ and the output $y$ are related by the linear equation $y = mx + c$, in which $m$ and $c$ are constants, is not a linear system (why?). A nonlinear system is one in which the input-output relation is such that the principle of superposition is not valid. In reality, a watershed is a nonlinear system, as the runoff from the watershed due to a storm is a nonlinear function of the (storm) rainfall over its area.

**Time Variant and Time Invariant Systems**

In a time invariant system, the input-output relationship does not depend on the time of application of the input, i.e. the output is the same for the same input at all times. The unit hydrograph in hydrology is a linear time invariant system.

**Continuous, Discrete and Quantized Systems**

In a continuous system, the changes in the system take place continuously; whereas in a discrete system, the state of the system changes at discrete intervals of time. A variable, input or an output, is said to be quantized when it changes only at certain discrete intervals of time and holds a constant value between intervals (e.g. rainfall record).

**Lumped Parameter and Distributed Parameter Systems**

A lumped parameter is one whose variation in space is either nonexistent or ignored (e.g. average rainfall over a watershed). A parameter is said to be a distributed one if its variation in one or more spatial dimensions is taken into account. The parameters of the system, the input or the output may be lumped. A lumped parameter system is governed by ordinary differential equations (with time as an independent variable), whereas a distributed parameter system is governed by partial differential equations (with spatial coordinates as independent variables). For example, a homogeneous isotropic aquifer is analyzed as a lumped parameter system. Instead, if the spatial variation of the transmissivity in modelling a water table aquifer is to be taken into account, the aquifer has to be modelled as a distributed parameter system.
Deterministic and Probabilistic Systems
In a deterministic system, if the input remains the same, the output remains the same, i.e. the same input will always produce the same output. The input itself may be deterministic or stochastic. In a probabilistic system, the input–output relationship is probabilistic rather than deterministic. The output corresponding to a given input will have a probability associated with it.

Stable Systems
A stable system is one in which the output is bounded if the input is bounded. Virtually all systems in hydrology and water resources are stable systems.

1.3 SYSTEMS APPROACH
The input–output relationship of a system is controlled by the nature, parameters of the system and the physical laws governing the system. In many of the systems in practice, the nature and the principal laws are very complex, and systems modelling in such cases uses simplifying assumptions and transformation functions, which convert the input to the corresponding output, ignoring the mechanics of the physical processes involved in the transformation. This requires conceptualization of the system and its configuration to be able to construct a mathematical model of it in which the input-output relationships are established through operating the system in a defined fashion. The specification of the system operation is what we refer to as the operating policy.

1.4 SYSTEMS ANALYSIS
Systems analysis may be said to be a formalization of the operation of the total system with all of its subsystems together. Systems analysis is usually understood as a set of mathematical planning and design techniques, which includes some formal optimization procedure. When scarce resources must be used effectively, systems analysis techniques stand particularly promising (for example, in optimal crop water allocation to several competing crops, under conditions of limited water supply). It must be clearly understood that systems analysis is not merely an exercise in mathematical modelling but spans much farther into processes such as design and decision. The techniques may use both descriptive as well as prescriptive models. The descriptive models deal with the way the system works, whereas the prescriptive ones are aimed at deciding how the system should be operated to best achieve the specified objectives.

1.4.1 Basic Problems in Systems Analysis
Basically there are two types of problems: analysis and synthesis. The first one is essentially a problem of prediction or a direct problem, whereas the second is a problem of identification or an inverse problem. In a prediction problem, we are required to determine the output knowing the input and the system operation. In an inverse problem, we are required to find the system (parameters), given the input and the output. In synthesis, we devise a model (system) that
will convert a known input to its corresponding known output. Here, we have to keep in mind the nature of the system and its operation.

1.4.2 Example Problems

**Prediction** In surface water hydrology, the problem is to predict the storm runoff (output), knowing the rainfall excess (input) and the unit hydrograph (system). In ground water hydrology, the problem is to determine the response (output) of a given aquifer (system), for given rainfall and irrigation application (input). In a reservoir (system) the problem is to determine irrigation allocations (output) for given inflow and storage (input), based on known or given operating policy.

**Identification** In surface water hydrology, the problem is to derive the unit hydrograph (system) given precipitation (input) and runoff data (output) for the concurrent period. It is assumed that all the complexities of the watershed including the geometry and physical processes in the runoff conversion are described by the unit hydrograph. In ground water hydrology, the problem is to determine the aquifer parameters (system), given the aquifer response (output) for known rainfall and irrigation application (input).

In a reservoir system, the problem is to determine the reservoir release policy (system operation) for a specified objective (output) for given inflows (input).

**Synthesis** The problem of synthesis is even more complex than the inverse problem mentioned earlier. Here no record of input and output are available. An example is the derivation of Snyder’s synthetic unit hydrograph using watershed characteristics to convert known values of rainfall excess to runoff.

1.4.3 Techniques of Water Resources Systems Analysis

The basic techniques used in water resources systems analysis are optimization and simulation. Whereas optimization techniques are meant to give global optimum solutions, simulation is a trial and error approach leading to the identification of the best solution possible. The simulation technique cannot guarantee global optimum solution; however, solutions, which are very close to the optimum, can be arrived at using simulation and sensitivity analysis.

Optimization models are embodied in the general theory of mathematical programming. They are characterized by a mathematical statement of the objective function, and a formal search procedure to determine the values of decision variables, for optimizing the objective function. The principal optimization techniques are:

1. **Linear Programming** The objective function and the constraints are all linear. It is probably the single-most applied optimization technique the world over. In integer programming, which is a variant of linear programming, the decision variables take on integer values. In mixed integer programming, only some of the variables are integers.
2. **Nonlinear Programming** The objective function and/or (any of) the constraints involve nonlinear terms. General solution procedures do not exist. Special purpose solutions, such as quadratic programming, are available for limited applications. However, linear programming may still be used in some engineering applications, if a nonlinear function can be either transformed to a linear function, or approximated by piece-wise linear segments.

3. **Dynamic Programming** Offers a solution procedure for linear or nonlinear problems, in which multistage decision-making is involved.

   The choice of technique for a given problem depends on the configuration of the system being analyzed, the nature of the objective function and the constraints, the availability and reliability of data, and the depth of detail needed in the investigation. Linear programming (LP), and dynamic programming (DP) are the most common mathematical programming models used in water resources systems analysis. Simulation, by itself, or in combination with LP, DP, or both LP and DP is used to analyze complex water resources systems.

**REFERENCE**


**Further Reading**

In most engineering problems, decisions need to be made to optimize (i.e., minimize or maximize) an appropriate physical or economic measure. For example, we may want to design a reservoir at a site with known inflows for meeting known water demands, such that the reservoir capacity is minimum, or we may be interested in locating wells in a region such that the aquifer drawdown is minimum for a given pumping pattern. Most engineering decision-making problems may be posed as optimization problems. In general, an optimization problem consists of:

(a) an objective function, which is a mathematical function of decision variables, that needs to be optimized, and
(b) a set of constraints that represents some physical (or other) conditions to be met.

The decision variables are the variables for which decisions are required such that the objective function is optimized subject to the constraints. In the first example mentioned, the reservoir capacity is one of the decision variables and the reservoir mass balance defines one set of constraints. In the second example, the locations—in terms of coordinates—of the wells are the decision variables, and again, the mass balance forms a set of constraints.

A general optimization problem may be expressed mathematically as

Maximize \( f(X) \)

subject to

\[ g_j(X) \leq 0 \quad j = 1, 2, \ldots, m \]

where, \( X \) is a vector of decision variables, \( X = [x_1, x_2, x_3, \ldots, x_n] \). In this general problem there are \( n \) decision variables (viz. \( x_1, x_2, x_3, \ldots, x_n \)) and \( m \) constraints. The complexity of the problem varies depending on the nature of the function \( f(X) \), the constraint functions \( g_j(X) \) and the number of variables and constraints.

Simulation is a technique by which we imitate the behavior of a system. Typically, we use simulation to answer ‘what-if’ type of questions. As against
optimization, where we are typically looking for the ‘best possible’ solution, in simulation we simply look at the behavior of the system for given sets of inputs. Simulation is a very powerful technique in analyzing most complex water resource systems in detail for performance evaluation, while optimization models yield results helpful in planning and management of large systems. In some situations, optimization models cannot be even applied due to computational limitations. In many situations, however, decision-makers would be interested in examining a number of scenarios rather than just looking at one single solution that is optimal. Typical examples where simulation is used in water resources include:

(a) analysis of river basin development alternatives,
(b) multireservoir operation problems,
(c) generating trade-offs of water allocations among various uses such as hydropower, irrigation, industrial and municipal use, etc., and
(d) conjunctive use of surface and ground water resources.

It may be noted that by repeatedly simulating the system with various sets of inputs it is possible to obtain near-optimal solutions.

In this chapter, an introduction to classical optimization using calculus is presented first (Section 2.1), followed by the two most commonly used optimization techniques, Linear Programming (Section 2.2) and Dynamic Programming (Section 2.3). An introduction to simulation is provided next (Section 2.4). Applications of these techniques to water resources problems are discussed in Part 2 of the book.

2.1 OPTIMIZATION USING CALCULUS

Some basic concepts and rules of optimization of a function of a single variable and a function of multiple variables are presented in this chapter.

2.1.1 Function of a Single Variable

Let \( f(x) \) be a function of a single variable \( x \), defined in the range \( a < x < b \) (Fig. 2.1).

![Function of a Single Variable](image)

**Local Maximum** The function \( f(x) \) is said to have a local maximum at \( x_1 \) and \( x_4 \), where it has a value higher than that at any other value of \( x \) in the neighborhood of \( x_1 \) and \( x_4 \). The function is a local maximum at \( x_1 \), if

\[
 f(x_1 - \Delta x_1) < f(x_1) < f(x_1 + \Delta x_1)
\]
**Basics of Systems Techniques**

**Local Minimum** The function \( f(x) \) is said to have a local minimum at \( x_2 \) and \( x_5 \), where it has a value lower than that at any other value of \( x \) in the neighborhood of \( x_2 \) and \( x_5 \).

The function is a local minimum at \( x_2 \), if
\[
f(x_2 - \Delta x) > f(x_2) < f(x_2 + \Delta x)
\]

**Saddle Point** The function has a saddle point at \( x_3 \), where the value of the function is lower on one side of \( x_3 \) and higher on the other, compared to the value at \( x_3 \). The slope of the function at \( x_3 \) is zero.
\[
f(x_3 - \Delta x) < f(x_3) < f(x_3 + \Delta x); \text{ slope of } f(x) \text{ at } x = x_3 \text{ is zero}
\]

**Global Maximum** The function \( f(x) \) is a global maximum at \( x_4 \) in the range \( a < x < b \), where the value of the function is higher than that at any other value of \( x \) in the defined range.

**Global Minimum** The function \( f(x) \) is a global minimum at \( x_2 \) in the range \( a < x < b \), where the value of the function is lower than that at any other value of \( x \) in the defined range.

**Convexity**

A function is said to be strictly convex, if a straight line connecting any two points on the function lies completely above the function. Consider the function \( f(x) \) in Fig. 2.2.

![Convex Function](image)

\( f(x) \) is said to be convex if the line \( AB \) is completely above the function (curve \( AB \)). Note that the value of \( x \) for any point in between \( A \) and \( B \) can be expressed as \( \alpha x_1 + (1 - \alpha)x_2 \), for some value of \( \alpha \), such that \( 0 \leq \alpha \leq 1 \).

Therefore,
\[
f(\alpha x_1 + (1 - \alpha)x_2) < \alpha f(x_1) + (1 - \alpha) f(x_2); \quad \text{where } 0 \leq \alpha \leq 1
\]

1. If the inequality sign \(<\) is replaced by \(\leq\) sign, then \( f(x) \) is said to be convex, but not strictly convex.

2. If the inequality sign \(<\) is replaced by \(=\) sign, \( f(x) \) is a straight line and satisfies the condition for convexity mentioned in 1 above. Therefore, a straight line is a convex function.
3. If a function is strictly convex, its slope increases continuously, or \[ \frac{d^2 f}{dx^2} > 0 \]. For a convex function, however, \[ \frac{d^2 f}{dx^2} \geq 0 \].

**Concavity**

A function is said to be strictly concave if a straight line connecting any two points on the function lies completely below the function. Consider the function, \( f(x) \), in Fig. 2.3.

![Concave Function](image)

**Fig. 2.3** Concave Function

A function \( f(x) \) is strictly concave, if the line \( AB \) connecting any two points \( A \) and \( B \) on the function lies completely below the function. (curve \( AB \)).

1. If the inequality \( > \) is replaced by \( \geq \), then \( f(x) \) is said to be concave, but not strictly concave.
2. If the inequality \( > \) is replaced by \( = \) sign, then \( f(x) \) is a straight line still satisfying the condition for concavity. Therefore a straight line is a concave function.
3. If a function is strictly concave, its slope decreases continuously, or \[ \frac{d^2 f}{dx^2} < 0 \]. For a concave function, however, \[ \frac{d^2 f}{dx^2} \leq 0 \].

It may be noted that a straight line is both convex and concave, and is neither strictly convex nor strictly concave.

A local minimum of a convex function is also its global minimum.
A local maximum of a concave function is also its global maximum.

The sum of (strictly) convex functions is (strictly) convex.
The sum of (strictly) concave functions is (strictly) concave.

If \( f(x) \) is a convex function, \(-f(x)\) is a concave function.
If \( f(x) \) is a concave function, \(-f(x)\) is a convex function.

In general, if \( f(x) \) is a convex function, and \( \alpha \) is a constant,
\[ \alpha f(x) \] is convex if \( \alpha > 0 \) and
\[ \alpha f(x) \] is concave if \( \alpha < 0 \).
2.1.2 Optimization of a Function of a Single Variable

The point at which a function will have a maximum or minimum is called a stationary point. A stationary point is a value of the independent variable at which the slope of the function is zero.

\[ x = x_0 \] is a stationary point if \( \frac{df}{dx}\big|_{x_0} = 0 \). This is a necessary condition for \( f(x) \) to be a maximum or minimum at \( x_0 \).

Sufficiency condition is examined as follows:

1. If \( \frac{d^2f}{dx^2} > 0 \) for all \( x \), \( f(x) \) is convex and the stationary point is a global minimum.

2. If \( \frac{d^2f}{dx^2} < 0 \) for all \( x \), \( f(x) \) is concave and the stationary point is a global maximum.

3. If \( \frac{d^2f}{dx^2} = 0 \), we should investigate further.

   In case of 3, find the first nonzero higher order derivative. Let this be the derivative of \( n \)th order.

   Thus, at the stationary point, \( x = x_0 \),

   \[ \frac{d^n f}{dx^n} = 0 \]

1. If \( n \) is even, \( x_0 \) is a local minimum or a local maximum.

   - If \( \frac{d^n f}{dx^n}\big|_{x_0} > 0 \), \( x_0 \) is a local minimum

   - If \( \frac{d^n f}{dx^n}\big|_{x_0} < 0 \), \( x_0 \) is a local maximum

2. If \( n \) is odd, \( x_0 \) is a saddle point.

2.1.3 Function of Multiple Variables

Let \( f(X) \) be a function of \( n \) variables represented by the vector \( X = (x_1, x_2, x_3, \ldots, x_n) \). Before coming to the criteria for convexity and concavity of a function of multiple variables, we should know the Hessian matrix (or H-matrix, as it is sometimes referred to) of the function. The Hessian matrix, \( H[f(X)] \), of the function, \( f(X) \), is defined as
The convexity and concavity of a function of multiple variables is determined by an examination of the eigen values of its Hessian matrix.

The eigen values of $H[f(X)]$ are given by the roots of the characteristic equation,

$$|I - H[f(X)]| = 0$$

where $I$ is an identity matrix, and $\lambda$ is the vector of eigen values. The function $f(X)$ is said to be positive definite if all its eigen values are positive, i.e. all the values of $\lambda$ should be positive. Similarly, the function $f(X)$ is said to be negative definite if all its eigen values are negative, i.e. all the values of $\lambda$ should be negative.

**Convexity and Concavity** If all eigen values of the Hessian matrix are positive, the function is strictly convex.

If all the eigen values of the Hessian matrix are negative, the function is strictly concave.

If some eigen values are positive and some negative, or if some are zero, the function is neither strictly convex nor strictly concave.

**2.1.4 Optimization of a Function of Multiple Variables**

**Unconstrained Optimization**

Let $f(X)$ be a function of multiple variables, $X = (x_1, x_2, x_3, ..., x_n)$.

A necessary condition for a stationary point $X = X_0$ is that each first partial derivative of $f(X)$ should equal zero.

$$\frac{df}{dx_1} = \frac{df}{dx_2} = ... = \frac{df}{dx_n} = 0.$$  

Whether the function is a minimum or maximum at $X = X_0$ depends on the nature of the eigen values of its Hessian matrix evaluated at $X_0$.

1. If all eigen values are positive at $X_0$, $X_0$ is a local minimum. If all eigen values are positive for all possible values of $X$, then $X_0$ is a global minimum.

2. If all eigen values are negative at $X_0$, $X_0$ is a local maximum. If all eigen values are negative for all possible values of $X$, then $X_0$ is a global maximum.

3. If some eigen values are positive and some negative or some are zero, then $X_0$ is neither a local minimum nor a local maximum.
Examine the following functions for convexity/concavity and determine their values at the extreme points.

1. \( f(X) = x_1^2 + x_2^2 - 4x_1 - 2x_2 + 5 \)

First determine the Hessian Matrix.

\[
\frac{\partial^2 f}{\partial x_1^2} = 2x_1 - 4; \quad \frac{\partial^2 f}{\partial x_2^2} = 2
\]

\[
\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1} = 0
\]

Therefore, \( H f(X) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \)

Eigen values of \( H \) are obtained by

\[
|I - H| = 0
\]

i.e.

\[
\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 0
\]

or

\[
(\lambda - 2)^2 = 0
\]

The eigen values are \( \lambda_1 = 2, \lambda_2 = 2 \).

As both the eigen values are positive, the function is a convex function (strictly convex). Also, as the eigen values do not depend on the value of \( x_1 \) or \( x_2 \), the function is strictly convex.

The stationary points are given by solving

\[
\frac{\partial f}{\partial x_1} = 2x_1 - 4 = 0
\]

and

\[
\frac{\partial f}{\partial x_2} = 2x_2 - 2 = 0
\]

i.e. \( x_1 = 2, x_2 = 1 \) or \( X = (2, 1) \).

Therefore the function \( f(X) \) has a global minimum at \( X = (2, 1) \).

2. \( f(X) = -x_1^2 - x_2^2 - 4x_1 - 8 \)

\[
\frac{\partial f}{\partial x_1} = -2x_1 - 4; \quad \frac{\partial^2 f}{\partial x_1^2} = -2
\]

\[
\frac{\partial f}{\partial x_2} = -2x_2; \quad \frac{\partial^2 f}{\partial x_2^2} = -2
\]
The Eigen values are \( \lambda_1= -2, \lambda_2 = -2 \).

Both Eigen values of H Matrix are negative and are independent of the value of \( x_1 \) and \( x_2 \). Therefore \( f(X) \) is a strictly concave function.

Stationary Point:
\[
\frac{\partial f}{\partial x_1} = -2x_1 - 4 = 0, \quad x_1 = -2
\]
\[
\frac{\partial f}{\partial x_2} = -2x_2 = 0, \quad x_2 = 0
\]

That is \( X = (-2, 0) \) and is a global maximum.

The function \( f(X) \) has a global maximum at \( X = (-2, 0) \) equal to -4.

3. \( f(X) = x_1^2 + x_2^2 - 3x_1 - 12x_2 + 20 \)

\[
\frac{\partial f}{\partial x_1} = 3x_1^2 - 3 \quad \frac{\partial^2 f}{\partial x_1^2} = 6x_1
\]

\[
\frac{\partial f}{\partial x_2} = 3x_2^2 -12 \quad \frac{\partial^2 f}{\partial x_2^2} = 6x_2
\]

\[
\frac{\partial^2 f}{\partial x_1 \partial x_2} = 0 \quad \frac{\partial^2 f}{\partial x_2 \partial x_1} = 0
\]

\[
H = \begin{bmatrix} 6x_1 & 0 \\ 0 & 6x_2 \end{bmatrix}
\]

Eigen values are given by the equation
\[
|\lambda I - H| = 0 \quad \text{or} \quad \begin{bmatrix} \lambda - 6x_1 & 0 \\ 0 & \lambda - 6x_2 \end{bmatrix} = 0
\]

or \( (\lambda - 6x_1)(\lambda - 6x_2) = 0 \)

Therefore \( \lambda_1 = 6x_1, \lambda_2 = 6x_2 \)

That is, if both \( x_1 \) and \( x_2 \) are positive, then both eigen values are positive, and \( f(X) \) is convex; or if both \( x_1 \) and \( x_2 \) are negative, then both eigen values are negative, and \( f(X) \) is concave.
Basics of Systems Techniques

Stationary points:

\[ \frac{\partial f}{\partial x_1} = 3x_1^2 - 3 = 0, \quad x_1 = \pm 1 \]

\[ \frac{\partial f}{\partial x_2} = 3x_2^2 - 12 = 0, \quad x_2 = \pm 2 \]

Therefore:

(i) \( f(X) \) has a local minimum at \((x_1, x_2) = (1, 2)\), equal to \(1 + 3 + (-2)^3 - 12(2) + 20 = 2\).

\( f_{\text{min}}(X) = 2 \) at \( X = (1, 2) \).

(ii) \( f(X) \) has a local maximum at \((x_1, x_2) = (-1, -2)\) equal to \((-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20 = 38\).

\( f_{\text{max}}(X) = 38 \) at \( X = (-1, -2) \).

At the points \((1, -2)\) and \((-1, 2)\), the function is neither convex nor concave. They are saddle points.

Constrained Optimization

We shall discuss in this section the conditions under which a function of multiple variables will have a local maximum or a local minimum, and those under which its local optimum also happens to be its global optimum. Let us first consider a function with equality constraints.

1. Function \( f(X) \) of \( n \) Variables with a Single Equality Constraint

Maximize or Minimize \( f(X) \)

Subject to \( g(X) = 0 \)

Note that \( f(X) \) and \( g(X) \) may or may not be linear.

We shall write down the Lagrangean of the function \( f(X) \) denoted by \( Lf(X, \lambda) \), and apply the Lagrangean multiplier method.

\[ Lf(X) = f(X) - \lambda g(X), \text{ where } \lambda \text{ is a Lagrangean multiplier.} \]

When \( g(X) = 0 \), optimizing \( Lf(X) \) is the same as optimizing \( f(X) \). The original problem of constrained optimization is now transformed into an unconstrained optimization problem (through the introduction of an additional variable, the Lagrangean multiplier).

If more than one equality constraint is present in the problem, the Lagrangean function of \( f(X) \) in this case is

\[ Lf(X, \lambda) = f(X) - \lambda_1 g_1(X) - \lambda_2 g_2(X) - \ldots - \lambda_m g_m(X) \]

where \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \).

A necessary condition for the function to have a maximum or minimum is that the first partial derivatives of the function \( L \) should be equal to zero,

\[ \frac{dL}{d\lambda_i} = 0, \quad i = 1, 2, \ldots, n \]

\[ \frac{dL}{d\lambda_p} = 0, \quad p = 1, 2, \ldots, m \]
The \((n + m)\) simultaneous equations are solved to get a solution, \((X_0, \lambda_0)\).

Let the second partial derivatives be denoted by

\[
k_{ij} = \frac{\partial^2 L}{\partial x_i \partial x_j}, \text{ evaluated at } X_0 \text{ for } i = 1, 2, \ldots, n; j = 1, 2, \ldots, n
\]

\[
h_p = \frac{\partial f_p(X)}{\partial x_i}, \text{ evaluated at } X_0, p = 1, 2, \ldots, m
\]

The sufficiency condition is specified below.

Consider the determinant \(D\), denoted as \(|D|\), given by

\[
|D| = \begin{vmatrix}
  k_{11} - \mu & k_{12} & \cdots & k_{1n} & h_1 & \cdots & h_n \\
  k_{21} & k_{22} - \mu & \cdots & k_{2n} & h_1 & \cdots & h_n \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  k_{n1} & k_{n2} & \cdots & k_{nn} - \mu & h_1 & \cdots & h_n \\
  h_1 & h_2 & \cdots & h_n & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  h_1 & h_2 & \cdots & h_n & 0 & \cdots & 0 
\end{vmatrix}
\]

This is a polynomial in \(\mu\) of order \((n - m)\) where \(n\) is the number of variables and \(m\) is the number of equality constraints. If each root of \(\mu\) in the equation \(|D| = 0\) is negative, the solution \(X_0\) is a local maximum. If each root is positive, then \(X_0\) is a local minimum. If some roots are positive and some negative, \(X_0\) is neither a local maximum nor a local minimum. Also, if all the roots are negative and independent of \(X\), then \(X_0\) is the global maximum. If all the roots are positive and independent of \(X\), then \(X_0\) is the global minimum.

**Example 2.1.2**

Maximize \(f(X) = -x_1^2 - x_2^2\)

Subject to \(x_1 + x_2 = 4\) or \(x_1 + x_2 - 4 = 0\)

Solution:

\(g(x) = x_1 + x_2 - 4 = 0\)

The Lagrangean is

\(Lf(X, \lambda) = -x_1^2 - x_2^2 - \lambda (x_1 + x_2 - 4)\)

At the stationary point,

\[
\frac{\partial L}{\partial x_1} = -2x_1 - \lambda = 0
\]

\[
\frac{\partial L}{\partial x_2} = -2x_2 - \lambda = 0
\]

\[
\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 - 4) = 0
\]
These equations yield $x_1 = x_2 = 2, \lambda = -4$.
Now we shall determine if this is a maximum.

$$k_{11} = \frac{\partial^2 L}{\partial x_1^2} = -2, \quad k_{12} = \frac{\partial^2 L}{\partial x_1 \partial x_2} = 0$$

$$k_{21} = \frac{\partial^2 L}{\partial x_2 \partial x_1} = 0, \quad k_{22} = \frac{\partial^2 L}{\partial x_2^2} = -2$$

$$h_{11} = \frac{\partial g}{\partial x_1} = 1, \quad h_{12} = \frac{\partial g}{\partial x_2} = 1$$

<table>
<thead>
<tr>
<th>$\Delta x$</th>
<th>$-2 - \mu$</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$-2 - \mu$</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

or $2\mu + 4 = 0$ giving $\mu = -2$.
As the only root is negative, the stationary point $X = (2, 2)$ is a local maximum of $f(X)$ and $f_{\text{max}}(X) = -8$.

2. **Function $f(X)$ with Inequality Constraints** An inequality constraint can be converted to an equality constraint by introducing an additional variable on the left-hand side of the constraint.

Thus a constraint $g(X) \leq 0$ is converted as $g(X) + s^2 = 0$, where $s^2$ is a non-negative variable (being square of $s$). Similarly, a constraint $g(X) \geq 0$ is converted as $g(X) - s^2 = 0$.

The solution is found by the Lagrangean multiplier method, as indicated, treating $s$ as an additional variable in each inequality constraint.

When the Lagrangean of $f(X)$ is formed with either type of constraint, equating the partial derivative with respect to (w.r.t.) $s$ gives,

$\Delta s = 0$, meaning either $\lambda = 0$ or $s = 0$.

1. If $\lambda > 0, s = 0$. This means that the corresponding constraint is an equality constraint (binding constraint or active constraint).
2. If $s^2 > 0, \lambda = 0$. This means that the corresponding constraint is inactive or redundant.

**Kuhn-Tucker Conditions**

The conditions mentioned above lead to the statement of Kuhn-Tucker conditions. These conditions are necessary for a function $f(X)$ to be a local maximum or a local minimum. The conditions for a maximization problem are given below.

Maximize $f(X)$
subject to $g_j(X) \leq 0, j = 1, \ldots, m$.

The conditions are as follows:
The necessary and sufficient conditions for optimization of a function of multiple variables subject to a set of constraints are discussed below. Consider the problem

Maximize/Minimize $Z = f(X)$
subject to $g_i(X) \leq 0, i = 1, 2, \ldots, j$
$g_i(X) \geq 0, i = j + 1, \ldots, k$
$g_i(X) = 0, i = k + 1, \ldots, m.$

Introduce variables $s_i$ into the inequality constraints to make them equality constraints or equations. Let $S$ denote the vector with elements $s_i$.

The Lagrangean is

$$L(X, S, \lambda) = f(X) - \sum_{i=1}^{j} \lambda_i [g_i(X) + s_i^2] - \sum_{i=j+1}^{k} \lambda_i [g_i(X) - s_i^2] - \sum_{i=k+1}^{m} \lambda_i g_i(X)$$

where $\lambda_i$ is the Lagrangean multiplier associated with constraint $i$.

**Necessary Conditions for a Maximum or Minimum**  
The first partial derivatives of $L(X, S, \lambda)$ with respect to each variable in $X$, $S$ and $\lambda$ should be equal to zero. The solution for a stationary point $(X_0, S_0, \lambda_0)$ is obtained by solving these simultaneous equations. This is a necessary condition. Sufficiency is checked by the following conditions.

**Sufficiency Conditions for a Maximum**  
$f(X)$ should be a concave function.
$g_i(X)$ should be convex; $\lambda_i \geq 0, i = 1, 2, \ldots, j.$
$g_i(X)$ should be concave; $\lambda_i \leq 0, i = j + 1, \ldots, k.$
$g_i(X)$ should be linear; $\lambda_i$ unrestricted, $i = k + 1, \ldots, m.$

**Sufficiency Conditions for a Minimum**  
$f(X)$ should be a convex function.
$g_i(X)$ should be a convex function; $\lambda_i \geq 0, i = 1, \ldots, j.$
$g_i(X)$ should be a concave function; $\lambda_i \leq 0, i = j + 1, \ldots, k.$
$g_i(X)$ should be linear; $\lambda_i$ unrestricted, $i = k + 1, \ldots, m.$
**Note:** For a maximum or a minimum, the feasible space or the solution space should be a convex region. A constraint set \( g_i(X) \leq 0 \) defines a convex region, if \( g_i(X) \) is a convex function for all \( i \). Similarly, a region defined by a constraint set \( g_i(X) \geq 0 \) is a convex region, if \( g_i(X) \) is a concave function for all \( i \).

It is practically better to stick to one set of criteria, i.e. either for maximization or minimization. We shall follow the criteria for maximization in the following examples while testing the sufficiency criterion. For this purpose, we shall rewrite the given problem to the following form:

Maximize \( f(X) \)
subject to \( g_i(X) \leq 0 \)

We shall reiterate here that a linear function is both convex and concave.

**Example 2.1.1**

Minimize \( f(X) = x_1^2 + x_2^2 - 4x_1 - 4x_2 + 8 \)
Subject to \(-x_1 - 2x_2 + 4 \geq 0, \ 2x_1 + x_2 \leq 5 \).

Solution:

\[
\begin{align*}
\frac{\partial f}{\partial x_1} &= 2x_1 - 4, & \frac{\partial^2 f}{\partial x_1^2} &= 2 \\
\frac{\partial f}{\partial x_2} &= 2x_2 - 4, & \frac{\partial^2 f}{\partial x_2^2} &= 2 \\
\frac{\partial^2 f}{\partial x_1 \partial x_2} &= \frac{\partial^2 f}{\partial x_2 \partial x_1} = 0, \\
H f(X) &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \\
12I - 1B &= \begin{bmatrix} \lambda - 2 & 0 \\ 0 & \lambda - 2 \end{bmatrix} = 0; \\
\lambda_1 = 2 \ ; \ \lambda_2 = 2, \text{ both being positive.}
\end{align*}
\]

Thus \( f(X) \) is a convex function (strictly convex). Therefore the function \(-f(X)\) is concave and can be maximized.

First convert the problem to a form
Maximize \( F(X) \)
subject to \( g(X) \leq 0 \).

The original problem is rewritten as

Maximize \(-f(X)\) = \(-x_1^2 - x_2^2 + 4x_1 + 4x_2 - 8 \)
subject to \( x_1 + 2x_2 - 4 \leq 0, \ x_1 + 2x_2 - 4 + x_1^2 = 0 \)

\[
\begin{align*}
2x_1 + x_2 - 5 &\leq 0, \ or \ 2x_1 + x_2 - 5 + x_1^2 = 0 \\
L[-f(X)] &= -x_1^2 - x_2^2 + 4x_1 + 4x_2 - 8 - \lambda_1 (x_1 + 2x_2 - 4 + x_1^2) - \lambda_2 (2x_1 + x_2 - 5 + x_1^2)
\end{align*}
\]
\[
\frac{\partial L}{\partial \lambda_1} = -2x_1 + 4 - \lambda_1 - 2\lambda_2 = 0 \\
\frac{\partial L}{\partial \lambda_2} = -2x_2 + 4 - 2\lambda_1 - \lambda_2 = 0 \\
\frac{\partial L}{\partial x_1} = -2\lambda_1 x_1 = 0, \text{ i.e. either } \lambda_1 \text{ or } x_1 \text{ is zero} \\
\frac{\partial L}{\partial x_2} = -2\lambda_2 x_2 = 0, \text{ i.e. either } \lambda_2 \text{ or } x_2 \text{ is zero} \\
\frac{\partial L}{\partial \lambda_1} = -(x_1 + 2x_2 - 4 + s_1^2) = 0 \\
\frac{\partial L}{\partial \lambda_2} = -(2x_1 + x_2 - 5 + s_2^2) = 0 \\
\]

(i) Assuming \(\lambda_2 = 0\), \(s_1 = 0\); \(x_1 = 8/5\), \(x_2 = 6/5\) and \(\lambda_1 = 4/5 > 0\), \(s_2^2 = 3/5 > 0\).
Here the conditions for a maximum are satisfied. No violations.

(ii) Assume \(\lambda_1 = 0\) and \(\lambda_2 = 0\).
Then the simultaneous equations give
\[
x_1 = x_2 = 2; s_1^2 = -2 \text{ (not possible)} \\
s_2^2 = -1 \text{ (not possible)}.
\]
This is not a solution to the problem.

Similarly,

(iii) Assume \(\lambda_1 = 0\) and \(s_2 = 0\).
The equations to be solved are:
\[
-2x_1 + 4 - 2\lambda_2 = 0 \\
-2x_2 + 4 - \lambda_2 = 0 \\
x_1 + 2x_2 + s_1^2 = 4 \\
2x_1 + x_2 = 5
\]
These equations give \(x_1 = 3/2\), \(x_2 = 2\), \(s_1^2 = -3/2\) (not possible), \(\lambda_2 = 1/2 > 0\).
This is not a solution.

(iv) Assume \(s_1 = 0\), \(s_2 = 0\).
Then
\[
-2x_1 + 4 - \lambda_1 - 2\lambda_2 = 0 \\
-2x_2 + 4 - 2\lambda_1 - \lambda_2 = 0 \\
x_1 + 2x_2 = 4 \\
2x_1 + x_2 = 5
\]
These equations yield \(\lambda_1 = 4/3 > 0\), \(\lambda_2 = -2/3\) (negative). As \(\lambda_2 < 0\), this is not a solution for a maximum.

Hence solution (i) i.e. \(x_1 = 8/5\), \(x_2 = 6/5\) is the only solution to the problem.
Thus \(-f(X)\) is a maximum of \(-0.8\) at \((8/5, 6/5)\), or \(f(X)\) is a minimum of \(0.8\) at \(X = (8/5, 6/5)\).
Note: In a clear case like this, when \( f(X) \) is strictly convex or \(-f(X)\) is strictly concave and the solution set is convex (i.e. the constraint set is a convex region being bounded by linear functions), there is a unique solution.

That is, only a particular combination of \( \lambda \) and \( s \) yields the optimum solution. Thus, in a given trial in a problem such as Example 2.3, with two constraints: if \( \lambda_1 \) and \( \lambda_2 \) are assumed to be zero, then \( s_1^2 \) and \( s_2^2 \) should both be positive, if \( \lambda_2 \) and \( s_2 \) are assumed to be zero, then \( \lambda_1 \) and \( s_1^2 \) should both be positive, if \( s_1 \) and \( s_2 \) are assumed to be zero, then \( \lambda_1 \) and \( \lambda_2 \) should both be positive.

The first trial, which satisfies these conditions, will be the optimal solution to the problem, and the computations can stop there.

<table>
<thead>
<tr>
<th>Problems</th>
<th></th>
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</thead>
<tbody>
<tr>
<td>2.1.1 Maximize ( f(X) = -x_1^2 - x_2^2 ) subject to ( x_1 + x_2 = 4 ) ( 2x_1 + x_2 \geq 5 ) ( \text{(Ans: } f(X) \text{ is max at } (2, 2)) )</td>
<td></td>
</tr>
<tr>
<td>2.1.2 Minimize ( f(X) = 5x_1^2 + x_2^2 = 4 ) subject to ( x_2 - 4 \geq -4x_1 ) ( -x_2 + 3 \leq 2x_1 ) ( \text{(Ans: } f_{\text{min}}(X) = 9 \text{ at } X = (2/3, 5/3) )</td>
<td></td>
</tr>
<tr>
<td>2.1.3 Minimize ( f(X) = (x_1 - 2)^2 + (x_2 - 2)^2 ) subject to ( x_1 + 2x_2 \leq 3 ) ( 8x_1 + 5x_2 \geq 10 ) ( \text{(Ans: } f_{\text{min}}(X) = 9/5 \text{ at } (7/5, 4/5) )</td>
<td></td>
</tr>
<tr>
<td>2.1.4 Minimize ( f(X) = x_1^2 + x_2^2 - 4x_1 - x_2 ) subject to ( x_1 + x_2 \geq 2 ) ( \text{(Ans: } x_1 = 2, x_2 = 1/2, f_{\text{min}}(X) = -17/4) )</td>
<td></td>
</tr>
<tr>
<td>2.1.5 Optimize ( f(X) = -x_1^2 - x_2^2 + 4x_1 + 6x_2 ) subject to ( x_1 + x_2 \leq 2 ) ( -2x_1 + 12 - 3x_2 \geq 0 ) ( \text{(Ans: } x_1 = 1/2, x_2 = 3/2, f_{\text{max}}(X) = 17/2) )</td>
<td></td>
</tr>
<tr>
<td>2.1.6 Minimize ( f(X) = 5x_1^2 + 2x_2 - x_1x_2 ) subject to ( x_1 + x_2 \geq 3 ) ( \text{(Ans: } x_1 = 5/2, x_2 = 31/12, f_{\text{min}}(X) = 357/72) )</td>
<td></td>
</tr>
<tr>
<td>2.1.7 Optimize ( f(X) = -2x^2 + 3xy - 4y^2 + 18x ) subject to ( x + y \leq 7 ) ( \text{(Ans: } x = 113/22, y = 41/22, f_{\text{max}}(X) = 73.659) )</td>
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</table>
2.2 LINEAR PROGRAMMING

Linear programming may be classified as the most popular optimization technique used ever in water resources systems planning. Its popularity is partly because of the readily available software packages for problem solution, apart from its ability to screen large-scale water resources systems in identifying potential smaller systems for detailed modelling and analysis. Systems analysts find this tool extremely useful as a screening model for very large systems, and as a planning model to determine the design and operating parameters for a detailed operational study of a given system.

2.2.1 General Formulation and Prelude to Simplex Method

Linear programming (LP) is a scheme of solving an optimization problem in which both the objective function and the constraints are linear functions of decision variables. There are several ways of expressing a linear programming formulation, which lend themselves to solutions, with appropriate modifications to the original problem.

We shall illustrate here maximization problem in LP in its classical form first, and discuss variations later in the section.

Maximize \[ z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \]
subject to

\[ \begin{align*}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &\leq b_2 \\
\vdots & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n &\leq b_m \\
x_1, x_2, \ldots, x_n &\geq 0
\end{align*} \]

where,
- \( z \) is the objective function,
- \( x_1, x_2, \ldots, x_n \) are decision variables,
- \( c_1, c_2, \ldots, c_n \) are coefficients of \( x_1, x_2, \ldots, x_n \), respectively, in the objective function,
- \( a_{ij}, \ldots, a_{jk}, a_{ki}, \ldots, a_{kn} \) are coefficients in the constraints,
- \( b_1, b_2, \ldots, b_m \) are non-negative right hand side values.

Each of the constraints can be converted to an equation by adding a slack variable to the left hand side. The coefficient of this slack variable in the objective function will be zero.

Standard Form (Equality Constraints)

There are many standard forms in which an LP problem is expressed, and we shall follow the standard form with equality constraints, as given here, throughout the section.

Maximize \[ z = c_1x_1 + c_2x_2 + \ldots + c_nx_n + c_{n+1}x_{n+1} + \ldots + c_{m+n}x_{m+n} \]
subject to

\[ \begin{align*}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n + x_{n+1} &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n + x_{n+2} &\leq b_2 \\
\vdots & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n + x_{n+m} &\leq b_m \\
x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{m+n} &\geq 0
\end{align*} \]
where the variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ are called slack variables. The objective function is written including the slack variables with coefficients $c_{n+1} = c_{n+2} = c_{n+3} = \cdots = c_{n+m} = 0$.

In this standard form, we have a total of $n + m$ variables ($n$ decision variables + $m$ slack variables) and a constraint set of only $m$ equations. These equations can be solved uniquely for any set of $m$ variables if the remaining $n$ variables are set to zero.

For example, in the simplex method (an iterative method, discussed later in the section), the starting solution is chosen to be the one in which the decision variables $x_1, x_2, \ldots, x_n$ are assumed zero, so that the slack variable in each equality constraint equals the right hand side of the equation, i.e. $x_{n+1} = b_1, x_{n+2} = b_2, \ldots, x_{n+m} = b_m$, in the starting solution in the simplex method. Obviously, the objective function value for this starting solution is $z = 0$. Iterations are performed in the simplex method on this starting solution for better values of the objective function till optimality is reached.

Before discussing the simplex method, the graphical method to an LP problem in two variables is illustrated in the following example to gain some insight into the method. It may be noted that if the problem has more than two decision variables, the graphical method in two dimensions illustrated below cannot be used.

**Example 2.2.1**  Two crops are grown on a land of 200 ha. The cost of raising crop 1 is 3 unit/ha, while for crop 2 it is 1 unit/ha. The benefit from crop 1 is 5 unit/ha and from crop 2, it is 2 unit/ha. A total of 300 units of money is available for raising both crops. What should be the cropping plan (how much area for crop 1 and how much for crop 2) in order to maximize the total net benefits?

**Solution:**

The net benefit of raising crop 1 = $5 - 3 = 2$ unit/ha

The net benefit of raising crop 2 = $2 - 1 = 1$ unit/ha

Let $x_1$ be the area of crop 1 in hectares and $x_2$ be that of crop 2, and $z$, the total net benefit.

Then the net benefit of raising both crops is $2x_1 + x_2$. However, there are two constraints. One limits the total cost of raising the two crops to 300, and the other limits the total area of the two crops to 200 ha. These two are the resource constraints. Thus the complete formulation of the problem is

$$\text{maximize} \quad z = 2x_1 + x_2 \quad \text{(2.1)}$$

subject to

$$\begin{align*}
3x_1 + x_2 & \leq 300 \\
x_1 + x_2 & \leq 200 \\
x_1, x_2 & \geq 0
\end{align*} \quad \text{(2.2)}$$

Equation (2.1) is the objective function and Eqs (2.2) are the constraints. The non-negative constraints for $x_1$ and $x_2$ indicate that neither $x_1$ nor $x_2$ can physically be negative (area cannot be negative).
Graphical Method  First, the feasibility region for the constraint set should be mapped. To do this, plot the lines $3x_1 + x_2 = 300$, $x_1 + x_2 = 200$, along with $x_1 = 0$ and $x_2 = 0$ as in Fig. 2.4. The region bounded by the non-negativity constraints is the first quadrant in which $x_1 \geq 0$ and $x_2 \geq 0$. The region bounded by the constraint $3x_1 + 2x_2 \leq 300$ is the region $OCD$ (it is easily seen that since the origin $x_1 = 0$, $x_2 = 0$ satisfies this constraint, the region to the left of the line $CD$ in which the origin lies is the feasible region for this constraint). Similarly, the region $OAB$ is the feasible region for the constraint $x_1 + x_2 \leq 200$.

Thus the feasible region for the problem taking all constraints into account is $OAPD$, where $P$ is the point of intersection of the lines $AB$ and $CD$. Any point within or on the boundary of the region, $OAPD$, is a feasible solution to the problem. The optimal solution, however, is that point which gives the maximum value of the objective function, $z$, within or on the boundary of the region $OAPD$.

Next, consider a line for objective function, $z = 2x_1 + x_2 = c$, for an arbitrary value $c$. The line shown in the figure is drawn for $c = 40$ and the arrows show the direction in which lines parallel to it will have higher value of $c$, i.e. if the objective function line is plotted for two different values of $c$, the line with a higher value of $c$ plots farther from the origin than the one with a lower value of $c$. We need to determine that value of $c$ (and therefore the values of $x_1$ and $x_2$ associated with it) corresponding to a line parallel to $2x_1 + x_2 = c$, farthest from the origin and at the same time passing through a point lying within or on the boundary of the feasible region. If the $z$ line is moved parallel to itself away from the origin, the farthest point on the feasible region that it touches is the point $P(50,150)$. This can be easily seen by an examination of the slopes of the $z$ line and the constraint lines.

Since the slope of the $z$ line is $-2$ which lies between $-3$ (slope of the line $3x_1 + x_2 = 300$) and $-1$ (slope of the line $x_1 + x_2 = 200$), the farthest point, in the feasible region away from the origin, lying on a line parallel to the $z$ line at $P$. Thus the point $P(x_1 = 50, x_2 = 150)$ presents the optimal solution to the
The maximized net benefit $z = Rs 250$. Let us note here that the optimal solution lies in one of the corners of the feasible region. In general, the optimum solution lies at one of the corner points of the feasible region or at a point on the boundary (of the feasible region). In the former case, the solution will be unique; in the latter it gives rise to multiple solutions (yielding the same optimum value of the objective function).

The graphical method can be used only with a two-variable problem. For a general LP problem, the most common method used is the simplex method.

### Prelude to Simplex Method

There are two important characteristics of the optimal solution to an LP problem. One is what we just saw in Example 2.2.1, that the optimal solution lies in one of the corners of the feasible region, or on its boundary. To see another significant feature of the solution to an LP problem, let us look upon the solution as one resulting from the following set of simultaneous equations:

\[
\begin{align*}
3x_1 + x_2 + x_3 &= 300 \tag{2.3} \\
x_1 + x_2 + x_4 &= 200 \tag{2.4} \\
x_1, x_2, x_3, x_4 &\geq 0
\end{align*}
\]

where $x_1$ and $x_4$ are slack variables in the respective constraints, Eq. (2.3) and Eq. (2.4), introduced to facilitate equality of the left and right hand sides.

Equations 2.3 and 2.4 have four variables in two equations. This can be uniquely solved when two of the variables $x_1, x_2, x_3$ and $x_4$ assume zero values. Our aim is to look for such a combination of $x_1, x_2, x_3$ and $x_4$ satisfying Eqs. 2.3 and 2.4, which make the objective function $z = 2x_1 + x_2 + 0x_3 + 0x_4$ a maximum. If we assume two of these variables to be zero, then the remaining two can be solved from the two equations. In general, if there are $m$ number of equality constraints and $n + m$ total number of variables (including slack variables), we can solve for any $m$ of the $n + m$ variables if we assign zero value to each of the remaining $n$ variables. In the starting solution for the simplex method, we assign zero values to the $n$ decision variables and the remaining $m$ variables are solved from the $m$ simultaneous equations. In the example problem, we now need a search procedure to determine the optimal combination of the four variables $x_1, x_2, x_3, x_4$ that maximizes the value of the objective function, $z$. This is done by iterating the starting solution to move to that adjacent corner point solution which results in the best value of $z$, in the simplex method. In any corner point solution, it may be noted that there can be at most $m$ number of nonnegative variables and at least $n$ number of zero-valued variables. This is the second important feature implicit in the simplex method.

### 2.2.2 Simplex Method

Before we begin to discuss the simplex method of solution for LP problems, we should get used to the following terminology.
Solution: A set of values assigned to the variables in a given problem is referred to as a solution. A solution, in general, may or may not satisfy any or all of the constraints.

Basis and basic variables: The basis is the set of basic variables. The number of basic variables is equal to the number of equality constraints. The variables in the basis only can be non-negative. The nonbasic variables are zeros. In the optimal solution of the example mentioned earlier, out of the total of four variables $x_1, x_2, x_3$, and $x_4$, the variables $x_1$ and $x_2$ are in the basis, and $x_3$ and $x_4$ are out of the basis, i.e., $x_1$ and $x_2$ are basic variables and $x_3$ and $x_4$ are non-basic variables in the optimal solution.

Non-basic variables: Variables which are outside (or not in) the basis are non-basic variables. $x_3$ and $x_4$ in the optimal solution of the example are non-basic variables.

Feasible solution: Any solution (set of values associated with each variable) that satisfies all the constraints is a feasible solution.

Infeasible solution: A solution which violates at least one of the constraints is an infeasible solution.

Basic solution: Assume there are a total number of $n + m$ variables ($n$ decision variables and $m$ slack variables) and a total number of $m$ equality constraints. Then a basic solution is one which has $m$ number of basic variables and $n$ number of non-basic variables. All non-basic variables are zeros. The basic solution will have at most $m$ non-zero variables and at least $n$ zero-valued variables.

Basic feasible solution: A basic solution which is also feasible is a basic feasible solution.

Initial basic feasible solution: The basic feasible solution used as an initial solution in the simplex method is called an initial basic feasible solution (this is the solution in which all the $n$ decision variables are set to zero).

In the simplex method, we start from an initial basic feasible solution (also referred to as the starting solution) and determine the optimal solution, iteratively.

The example problem (Example 2.2.1) will now be solved using the simplex method:

Maximize $Z = 2x_1 + x_2$
subject to (s.t.) $3x_1 + x_2 \leq 300$
$x_1 + x_2 \leq 200$
$x_1, x_2 \geq 0$

First introduce slack variables (non-negative) and convert the constraints into equality constraints.

Max $z = 2x_1 + x_2 + 0x_3 + 0x_4$
subject to $3x_1 + x_2 + x_3 = 300$
$x_1 + x_2 + x_4 = 200$
$x_1, x_2, x_3, x_4 \geq 0$
Here the total number of variables, \( n = 4 \), and the number of constraints, \( m = 2 \). Therefore two of these four variables have to be necessarily set to zero to enable us to solve the two equations to determine the remaining two variables. For example, consider the solution \((x_1, x_2, x_3, x_4) = (25, 25, 200, 150)\). This solution is a feasible solution, but not a basic solution (verify). On the other hand, the solution \((100, 25, 0, 0)\) is a basic solution but infeasible, whereas the solution \((100, 0, 0, 100)\) is a basic feasible solution (because it has at least two zero valued variables and the solution satisfies both constraints).

In principle, one can start from any basic feasible solution, but the easiest way to identify an initial basic feasible solution is to choose the two slack variables \(x_3\) and \(x_4\) as basic variables and \(x_1\) and \(x_2\) as nonbasic variables (which therefore assume zero values). Then the equations right away yield \(x_3 = 300\), and \(x_4 = 200\), the respective right hand side values. Note that the objective function \(z = 0\) for this solution.

We shall now start with the solution \((0, 0, 300, 200)\) as the initial basic feasible solution, or the starting solution for the simplex method. The next step is to iterate on this solution to get a better solution (yielding a better value of the objective function).

The procedure is explained by means of the simplex tableau, Table 2.1. For convenience, the objective function is also written as an equality constraint as follows:

\[
\begin{align*}
z - 2x_1 - x_2 &= 0 \\
2x_3 - x_4 &= 0
\end{align*}
\]

And in terms of all the four variables as follows:

\[
\begin{align*}
z - 2x_1 - x_2 - 0x_3 - 0x_4 &= 0 \\
2x_3 - x_4 &= 0
\end{align*}
\]

For purposes of computations using the simplex method, this equation is considered as a constraint, and in all iterations, \(z\) is taken as a basic variable.

Table 2.1 shows the basic variables under the column ‘Basis’. The last column ‘RHS’ in each row gives the value of the basic variable in the current solution. The elements in each row connecting these two columns are the coefficients of the variables in the constraint represented by that row. Two important features of this table are:

1. each basic variable appears in only one of the constraints with a coefficient unity, with zero coefficients in all other constraints, and
2. the coefficient in the \(z\)-row for each basic variable is zero. The value of \(z\) in the current solution \([X = (0, 0, 300, 200)]\) is zero.

**Iteration 1**

We now have to find another basic feasible solution (a corner point solution adjacent to the starting solution in the feasible region) which yields a higher value of the objective function. An adjacent corner point solution will have a new basis with one of the basic variables replaced by one of the nonbasic variables in the starting solution. Which variable is to be replaced by which depends on the requirement that \(z\) value should increase the maximum by this change.
Entering Variable

The entering variable, i.e., the variable entering the basis, is chosen as that nonbasic variable which has the most negative coefficient in the \( z \)-row, in this case \(-2\) under \( x_1 \). Thus \( x_1 \) is the entering variable. The most negative coefficient indicates that the entry of \( x_1 \), rather than that of \( x_2 \) (the other nonbasic variable), into the basis contributes to the increase in the objective function most. The column under the variable \( x_1 \) is identified as the pivotal column.

Departing Variable

One of the two existing basic variables (\( x_3 \) or \( x_4 \)) should make way (depart) to admit \( x_1 \) into the new basis. This is decided based on the criterion that the departing variable should allow the entry of \( x_1 \) to the maximum (as a higher value of \( x_1 \) means a higher value of \( z \)) without making any other variable negative in the new solution.

### Table 2.1 Starting Solution

<table>
<thead>
<tr>
<th>Basis</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>RHS</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 1</td>
<td>( x_1 )</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>300</td>
</tr>
<tr>
<td>Row 2</td>
<td>( x_4 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>200</td>
</tr>
<tr>
<td>Row z</td>
<td>( z )</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Entering variable

Iteration 1

<table>
<thead>
<tr>
<th>Basis</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>RHS</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 1</td>
<td>( x_1 )</td>
<td>1</td>
<td>1/3</td>
<td>1/3</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>Row 2</td>
<td>( x_4 )</td>
<td>0</td>
<td>2/3</td>
<td>-1/3</td>
<td>1</td>
<td>100</td>
</tr>
<tr>
<td>Row z</td>
<td>( z )</td>
<td>0</td>
<td>-1/3</td>
<td>2/3</td>
<td>0</td>
<td>200</td>
</tr>
</tbody>
</table>

Entering variable

### Iteration 2 (solution)

<table>
<thead>
<tr>
<th>Basis</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>RHS</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 1</td>
<td>( x_1 )</td>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>-1/2</td>
<td>50</td>
</tr>
<tr>
<td>Row 2</td>
<td>( x_2 )</td>
<td>0</td>
<td>1</td>
<td>-1/2</td>
<td>3/2</td>
<td>150</td>
</tr>
<tr>
<td>Row z</td>
<td>( z )</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>1/2</td>
<td>250</td>
</tr>
</tbody>
</table>

Identify the coefficients, which are strictly positive in the column of the entering variable, known as the pivotal column, except the objective function row, of the current solution. In the example, \( x_1 \) is the entering variable and its coefficients 3 and 1 in the two rows, Row 1 and Row 2, are positive. Compute the ratio of the RHS to the positive coefficient under the pivotal column for each row. Pick the row corresponding to the least of these ratios, which is 100 in this case. Mark that as the pivotal row. The basic variable, \( x_{30} \), in the current solution, corresponding to this row in the simplex table, will be the departing variable. The coefficient, which is common to the pivotal row and the pivotal
column, is the pivot coefficient or the pivot, which is 3, in this case. The ratio test needs to be applied only for those pivot column coefficients that are strictly positive.

The departing variable is that basic variable corresponding to Row $i$ for which the value

\[(\text{Ratio}) = \frac{b_i}{a_{ij}}\]

for $a_{ij} > 0$, is minimum, over all $i = 1, 2, \ldots, m$.

where $i$ is the row and $j$ is the pivot column (corresponding to the entering variable $x_j$).

It may be noted that if all the coefficients $a_{ij}$ of the entering variable in the pivot column $j$ are nonpositive, it means that the problem is ill posed (giving rise to an unbounded solution).

Thus, $x_1$ replaces $x_3$ in the new solution which has $(x_1, x_3)$ as the basis. However, the coefficients in the simplex table should be worked out for this new basis to conform to the two important features mentioned under “Prelude to simplex method” earlier. This is achieved by the Gauss-Jordan transformation.

Gauss-Jordan transformation: The new pivot row (Row 1) is obtained by dividing the elements of each old row by the pivot coefficient.

\[
\text{New pivot row} = \frac{\text{Old pivot row}}{\text{pivot coefficient}}
\]

The new pivot row thus is $(1 \ 1/3 \ 1/3 \ 0, \ 100)$. This is the new Row 1 (iteration 1). The rows other than the pivot row are transformed as follows in the iteration.

\[
\text{New row} = \text{old row} - (\text{pivot column coefficient}) \times (\text{New pivot row})
\]

The new Row 2 is obtained by deducting the product of the elements of the new pivot row and the pivot column coefficient (equal to 1) from the elements of the old Row 2. Thus the new Row 2 is given by

\[
\text{New Row 2} = [1 \ 1/3 \ 1/3 \ 0, \ 100] - [1 \ 1/3 \ 1/3 \ 0, \ 100] \\
= [0 \ 2/3 \ -1/3 \ 1, \ 100]
\]

Similarly the new z-row is computed,

\[
\text{New Row z} = [-2 \ -1 \ 0 \ 0] - (-2) \times [1 \ 1/3 \ 1/3 \ 0, \ 100] \\
= [0 \ -1/3 \ 2/3 \ 0, \ 200]
\]

This completes iteration 1. The objective function value increased to 200 in this iteration from 0 in the starting solution. This solution would have been optimal if all the coefficients of the z-row were non-negative. The solution in this iteration is not optimal as there is still one negative coefficient ($-1/3$) in the z-row (under $x_3$). Another iteration is therefore needed.

**Iteration 2**

Since there is only one negative coefficient in Row 2 under $x_2$, $x_2$ is the entering variable. The departing variable is determined the same way as before and happens to be $x_3$ in this case. Thus the basic variables in iteration 2 are $x_1$ and $x_2$. By repeating the row transformations as before, we find that the coefficients in the z-row in iteration 2 are all non-negative. The coefficients in the
z-row under the basic variables \( x_3 \) and \( x_4 \) will be zero anyway if the row computations are carried out correctly. What makes the solution in iteration 2 optimal is the non-negativity of the coefficients in the z-row under each of the nonbasic variables \( x_3 \) and \( x_4 \) (both in this case being equal to \( 1/2 \)). Thus, there is no further scope of increasing the objective function value beyond 250. The solution, therefore, is optimal with \( x_3^* = 50 \), \( x_4^* = 150 \) and \( z_{\text{max}} = 250 \).

In summary, the general procedure for the simplex method is as follows:

1. Express the given LP problem in the standard form, with equality constraints and non-negative right-hand side values.
2. Identify the starting solution and construct the simplex table.
3. Check for optimality of the current solution. The solution will be optimal if all the coefficients in the z-row are non-negative. Else, an iteration is needed.
4. Identify the entering variable. This is the nonbasic variable with the most negative coefficients in the z-row.
5. Identify the departing variable. This is the basic variable in the current solution in Row \( i \) for which
   \[
   \frac{b_i}{a_{ij}} > 0
   \]
   is minimum
   A test needs to be applied for those \( a_{ij} > 0 \) (strictly).
6. Perform the row transformation (Gauss-Jordan transformation) and get the new solution.
7. Repeat steps 3 to 6 until optimal solution is obtained.

**Cases for a Tie**

The entering variable: If two nonbasic variables have the same most-negative coefficient in the z-row in any iteration, there is a tie for the entering variable. The tie can be broken by arbitrarily choosing any one variable as the entering variable.

The departing variable: If the ratio (between the right-hand side value and the positive pivot column coefficient in a row) is the same for two or more rows, and is also the minimum among the ratios for all rows, there is a tie for the departing variable. Here also, any one variable can be arbitrarily selected as the departing variable. This results in a degenerate solution.

A degenerate solution is one in which at least one of the basic variables has zero value. This happens when, in an earlier iteration, there is a tie for the minimum ratio for determining the departing variable. Degeneracy reveals that there is at least one redundant constraint. In general, the simplex method results in the same sequence of iterations without improving the value of the objective function and without terminating the computations. This is referred to as 'cycling.' We shall not discuss more of this, as there is very little probability of encountering such a situation in practice.
Multiple Solutions

Multiple solutions are alternate solutions to a problem yielding the same optimal value of the objective function. The existence of an alternate optimal solution is indicated by the presence of a zero in the z-row under a nonbasic variable in the final simplex table giving the optimal solution. The alternate solution is obtained by choosing this nonbasic variable as the entering variable, and finding a new solution in the next iteration. Thus, two corner point solutions are determined, both giving the same optimal value for $z$. The solution for any point on the line (in the hyperplane) connecting these two corner points can be expressed as a linear combination of the two corner point solutions and represents an alternate solution to the problem. Thus a problem detected of having an alternate solution has, in fact, an infinite number of solutions, all giving the same optimal value for $z$.

Let $X_1 = (x_1, x_2, \ldots, x_n)$ and $X_2 = (s_1, s_2, \ldots, s_n)$ be two identified solutions to an LP problem that has a total number of variables $n$ (inclusive of slack variables). The alternate solutions are given by:

$$X = \alpha X_1 + (1 - \alpha) X_2$$

where $0 \leq \alpha \leq 1$, $\alpha$ being a scalar.

The solution $X$ is a linear combination of $X_1$ and $X_2$. By varying $\alpha$, one can generate an infinite number of solutions to the problem.

**Example 2.2.2 (Multiple solutions)**

Consider the problem

Maximize $z = 2x_1 + x_2$

subject to

$$3x_1 + x_2 \leq 300$$
$$4x_1 + 2x_2 \leq 500$$
$$x_1, x_2 \geq 0$$

Writing in the standard form, with $x_3, x_4$ as slack variables:

Maximize $z = 2x_1 + x_2$

subject to

$$3x_1 + x_2 + x_3 = 300$$
$$4x_1 + 2x_2 + x_4 = 500$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Table 2.2 shows the iterations required to arrive at the final simplex table, giving the optimal solution using the simplex method.

<table>
<thead>
<tr>
<th>Basis</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>RHS</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_3$</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>300</td>
<td>300/3 = 100</td>
</tr>
<tr>
<td>$x_4$</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>500</td>
<td>500/4 = 125</td>
</tr>
<tr>
<td>$z$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.2 Starting Solution

<table>
<thead>
<tr>
<th>Basis</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>RHS</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_3$</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>300</td>
<td>300/3 = 100</td>
</tr>
<tr>
<td>$x_4$</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>500</td>
<td>500/4 = 125</td>
</tr>
<tr>
<td>$z$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

**Table 2.2 Starting Solution**

- Departing variable
- Entering variable
### Iteration 1

<table>
<thead>
<tr>
<th>Basis</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>RHS</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>1/3</td>
<td>1/3</td>
<td>0</td>
<td>100</td>
<td>$100 + 1/3 = 300$</td>
</tr>
<tr>
<td>$x_4$</td>
<td>0</td>
<td>2/3</td>
<td>-4/3</td>
<td>1</td>
<td>100</td>
<td>$100 + 2/3 = 150$</td>
</tr>
<tr>
<td>$z$</td>
<td>0</td>
<td>-1/3</td>
<td>2/3</td>
<td>0</td>
<td>200</td>
<td></td>
</tr>
</tbody>
</table>

- **Entering variable:** $x_4$
- **Departing variable:** $x_2$

### Iteration 2

<table>
<thead>
<tr>
<th>Basis</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>RHS</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1/2</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>-2</td>
<td>3/2</td>
<td>150</td>
<td></td>
</tr>
<tr>
<td>$z$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>250</td>
<td></td>
</tr>
</tbody>
</table>

- **Entering variable:** $x_3$

### Iteration 3

<table>
<thead>
<tr>
<th>Basis</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>RHS</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1/2</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>$x_2$</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>250</td>
<td></td>
</tr>
<tr>
<td>$z$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>250</td>
<td></td>
</tr>
</tbody>
</table>

**Note:** In Iteration 2, the solution reached the optimal stage. However, the zero under the nonbasic variable $x_3$ in the z-row, indicates the existence of an alternate solution, which is found in Iteration 3 by admitting $x_3$ into the basis.

The two solutions are $(x_1, x_2) = (50, 150)$ in Iteration 2, and $(x_1, x_2) = (0, 250)$ in Iteration 3. Any point on the line joining these two corner points is a solution, as it can be expressed as a linear combination of the two solutions.

### Unbounded Solution

An unbounded solution is one in which the objective function value can be arbitrarily increased without violating the constraints. A very simple example is

Maximize \( z = x_1 + x_2 \)

subject to \( x_1 \geq 5, x_2 \leq 10; x_1, x_2 \geq 0 \).

Just by inspection, one would arrive at the solution, \( x_1 = \infty, x_2 = 10 \), and therefore \( z = \infty \) (unbounded).

In some LP problems, the constraints may be such that the feasible region may be unbounded at least in one direction. In this case, the solution space will be unbounded in that direction. The objective also may be unbounded if it can increase, in a maximization problem, with the unbounded variable. An unbounded solution space, i.e. an unbounded feasible region, can be recognized from the simplex table if, at any iteration, the pivot column coefficients of a nonbasic variable, in all rows other than the z-row, are non-positive (\( \leq 0 \)). In addition, if the coefficient of that variable in the objective function row is negative in a maximization problem, then the objective value is also unbounded.
2.2.3 Variations from the Standard Form

A given LP problem can be written in the standard form by making suitable modifications, if necessary.

Nature of the Objective If the problem is one of minimization, the negative of the objective function is maximized to get the same result.

Example

Minimize \( z = f(X) \) where \( X \) is a vector of decision variables and can be written as

Maximize \( (-z) = -f(X) \)

e.g. Minimize \( Z = 3x_1 + 4x_2 \) is the same as writing

Maximize \( (-Z) = -3x_1 - 4x_2 \)

Inequality Sign If a constraint in the original problem has \( \geq \) type inequality, the constraint can be written as an equation by introducing a surplus variable. For example:

\[ 3x_1 + 4x_2 \geq 12 \]

can be written as

\[ 3x_1 + 4x_2 - x_3 = 12, \]

where \( x_3 \) is a surplus variable.

The purpose of a surplus variable in a constraint with \( \geq \) sign is to convert the inequality constraint into an equality constraint.

Variables Unrestricted in Sign In the standard form of LP, all variables are specified to be non-negative. If a variable \( x \) is unrestricted in sign (i.e., it may take a positive or negative value), it can always be expressed as the difference between two non-negative variables,

\[ x = x_1 - x_2 \]

where \( x_1, x_2 \geq 0 \).

In the optimal solution, if \( x \) is positive, then \( x_2 \) will be zero so that \( x = x_1 \), and if negative, \( x_1 \) will be zero so that \( x = -x_2 \). Thus, \( x_1 - x_2 \) is substituted for \( x \) and the solution sought in the usual manner. The value of \( x \) is inferred from the optimal values of \( x_1 \) and \( x_2 \) in the final solution.

Right-hand Side Value, \( b \) In the standard form, all \( b_i \) are non-negative. A negative \( b \), such as in the constraint,

\[ 2x_1 + 3x_2 \leq -5, \]

is handled by adding an artificial variable to the equation.
can be set positive by writing the constraint as \(-2x_1 - 3x_2 \geq 5\) and then introducing a surplus variable \(x_3\) such that
\[-2x_1 - 3x_2 - x_3 = 5\]

Further, we need an artificial variable to determine the starting solution,
\[-2x_1 - 3x_2 - x_3 + \mu = 5\], where \(\mu\) is the artificial variable.

When, finally, the LP problem is written in the standard form (with equality constraints), the total number of variables includes decision variables, slack variables, surplus variables and artificial variables. The simplex method is used to determine the values of all these variables in the optimal solution. It is only then that the optimal values of all decision variables will be known, though the other variables are not present in the original LP problem.

2.2.4 The Dual Problem

A given LP problem may be termed a primal problem, in the sense that there is a dual problem associated with it. Every primal LP problem will have its dual. Sometimes, it is easier to formulate and solve the dual problem, rather than the primal problem, and thereby determine the solution of the primal. The solution of the dual is extremely handy if the primal problem has a small number of decision variables and a large number of constraints.

Methods of solving a primal problem with artificial variables are relatively cumbersome. The dual simplex method offers a method of solving the problem without having to introduce artificial variables. We shall illustrate this method later in this section. If one knows the primal simplex and the dual simplex methods, any LP problem (with any type of constraint inequality, and whatever the nature of optimization) can easily be solved by one of these two methods. In general, if there is a \(\geq\) type inequality constraint with non-negative right-hand side, the dual simplex method would be relatively easier method to get the solution as we do not have to deal with an artificial variable.

Before we illustrate the dual simplex method, it is necessary to understand the definition of the dual and its formulation for a given primal LP problem, and its structure.

The Dual

Let us consider our example primal problem:

\[
\begin{align*}
\text{Maximize} & \quad z = 2x_1 + x_2 \\
\text{subject to} & \quad 3x_1 + x_2 \leq 300 \\
& \quad x_1 + x_2 \leq 200 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

It should be noted, for the purpose of formulating the dual, that for every primal constraint, there is a dual variable; and for every primal variable, there is a dual constraint. Thus the above example will have two dual variables, say \(y_1\) and \(y_2\), corresponding to constraint 1, and \(y_2\) corresponding to constraint 2. As there are two variables in the primal, there will be two constraints in the dual, one each corresponding to \(x_1\) and \(x_2\). There are other properties in the
dual that help its formulation. The nature of optimization is reversed in the
dual, i.e. the dual is a minimization problem, if the primal is one of maximiza-
tion. The right-hand sides of the primal (in this case 300 and 200) form the
coefficients of the dual variables, corresponding to the respective primal con-
straints, in the objective function.
The objective function: \( z' \), for the dual of the example primal problem is thus:

\[
\text{Minimize } z' = 300y_1 + 200y_2
\]

The dual constraints are formed as follows:
The coefficients in a dual constraint (corresponding to a given primal vari-
able, say \( x_i \)) will be the coefficients of the primal variable, \( x_i \), in the respective
constraints of the primal. In the example problem, the dual constraint corre-
sponding to the primal variable \( x_1 \) will have coefficients 3 for \( y_1 \) (from the first
primal constraint) and 1 for \( y_2 \) (from the second primal constraint).
The inequality sign for a dual constraint depends on the nature of the primal
variable associated with it. In a maximization problem, if the primal variable is
non-negative, as in the example primal problem, the associated dual constraint
(for a minimizing objective in the dual) will be of \( \geq \) sign.
The right-hand side of a dual constraint associated with a primal variable is
equal to its coefficient in the objective function of the primal. Thus:

- Primal variable \( x_1 \): Dual constraint is: \( 3y_1 + y_2 \geq 2 \)
- Primal variable \( x_2 \): Dual constraint is: \( y_1 + y_2 \geq 1 \)

All the dual variables are non-negative for a primal maximization problem
with all \( \leq \) type constraints.

Thus, the dual of the example primal problem is

\[
\begin{align*}
\text{Minimize } & z' = 300y_1 + 200y_2 \\
\text{subject to } & 3y_1 + y_2 \geq 2 \\
& y_1 + y_2 \geq 1 \\
& y_1, y_2 \geq 0
\end{align*}
\]

Note that the procedure mentioned above for formulating the dual applies to
a problem of the example type given, i.e. with a maximization objective, non-
negative right-hand sides with all constraints of \( \leq \) type and all variables being
non-negative. If there is a change in any of these conditions in a given primal
problem, then there is a corresponding departure in the rules for formulating
the dual. For example, if a primal variable is unrestricted in sign, the corre-
sponding dual constraint will have an equality sign. On the other hand, if there
is an equality constraint in the primal, the corresponding dual variable will be
unrestricted in sign. If there are multiple variations in the primal problem, not
conforming to the type of primal problem mentioned earlier, a whole lot of
confusion could arise in writing the dual. Because of this, a general procedure
applicable to formulae the dual for all variations in the primal problem is
indicated below.

**General Procedure for Formulating the Dual** Let the primal be written in
the standard form (equality constraints) as follows:
Optimize (maximize or minimize) \( z = \sum_{j=1}^{n} e_j x_j \)

subject to
\[
\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, 2, \ldots, m
\]
\[
x_j \geq 0, \quad j = 1, 2, \ldots, n.
\]

Here, there are \( n \) variables, \( x_j \) including slack and surplus variables as may be required to express the primal in the above form. Note that all \( b_i \) should be non-negative.

1. There is a dual variable associated with each primal constraint. Thus there are \( m \) variables in the dual.
2. There is a dual constraint associated with each primal variable, \( x_j \). Thus there are \( n \) constraints in the dual.
3. The coefficients of a primal variable in the primal constraints form the coefficients of the dual variables in the dual constraint (corresponding to the primal variable).
4. The inequalities in the dual constraints are all of \( \geq \) sign for a maximization objective in the primal, and are of \( \leq \) sign for a minimization objective in the primal.
5. The right-hand sides of the primal constraints form the coefficients of the dual variables in the objective function of the dual.
6. The nature of objective function in the dual is reversed; that is, the dual objective is minimization if the primal objective is maximization, and vice versa.
7. All variables in the dual are unrestricted in sign.

Note that this constraint may be overridden by one or more of the other dual constraints in a problem.

From these rules, one can write the dual of any primal problem given in any form without confusion.

Consider the example primal problem:

Maximize \( z = 2x_1 + x_2 \)

subject to
\[
3x_1 + 2x_2 \leq 300
\]
\[
x_1 + x_2 \leq 200
\]
\[
x_1, x_2 \geq 0.
\]

There are four variables and two constraints (apart from the non-negative constraints). Therefore, the dual will have two variables and four constraints.

Let the dual variables be \( y_1 \) and \( y_2 \) corresponding to constraints (1) and (2) of the primal, respectively. The dual constraints corresponding to the four primal variables are:
The last constraint, that \( y_1 \) and \( y_2 \) are unrestricted, are overridden by the constraints \( y_1 \geq 0, y_2 \geq 0 \) and hence are redundant.

The objective is to minimize \( z' = 300y_1 + 200y_2 \).

Thus the dual of the example primal problem is

\[
\begin{align*}
\text{Minimize} \quad & z' = 300y_1 + 200y_2 \\
\text{subject to} \quad & 3y_1 + y_2 \geq 2 \\
& y_1 + y_2 \geq 1 \quad \text{or } y_1, y_2 \geq 0 \\
\end{align*}
\]

It can easily be seen that the dual of the dual is the primal itself.

### 2.2.5 Dual Simplex Method

There are some basic differences between the primal simplex and the dual simplex methods. The primal simplex method starts from a nonoptimal feasible solution and moves towards the optimal solution, maintaining feasibility every time. The dual simplex method starts with an infeasible basic solution (and therefore a ‘super’ optimal solution) and strives to achieve feasibility, while satisfying optimality criterion every time. In other words, in the dual simplex method, the earliest iteration that gives a feasible solution is the optimal solution sought, because optimality is maintained in each iteration. In contrast, in the primal simplex method, feasibility is ensured in each iteration, and the earliest iteration that meets the optimality criterion is the optimal solution sought.

The dual simplex method, like the primal simplex method, has rules for the entering variable, the departing variable and testing the feasibility of a solution.

The method will be illustrated by an example problem for maximization. If the objective is minimization, the objective function can be written in the negative form for maximization, and the method indicated herein may be applied.

Minimize \( z = 2x_1 + x_2 \)

or maximize \( -z = z' = -2x_1 - x_2 \)

subject to

\[
\begin{align*}
3x_1 + x_2 & \geq 3 \\
4x_1 + 3x_2 & \geq 6 \\
x_1 + 2x_2 & \leq 3 \\
x_1, x_2 & \geq 0 \\
\end{align*}
\]

Rewrite the constraints in \( \leq \) type.

\[
\begin{align*}
-3x_1 - x_2 & \leq -3 \\
-4x_1 - 3x_2 & \leq -6 \\
x_1 + 2x_2 & \leq 3 \\
x_1, x_2 & \geq 0 \\
\end{align*}
\]
Write each constraint as an equation by introducing a slack variable. The equality constraints may have negative right-hand sides as well, unlike in the primal LP problem in standard form. In fact, the dual simplex method requires at least one negative right-hand side value in order to be able to start from an infeasible solution!

The problem is rewritten as

\[ z' + 2x_1 + x_2 = 0 \] Row \( z \), Objective function row
\[-3x_1 - x_2 + x_3 = -3 \] Row 1
\[-4x_1 - 3x_2 + x_4 = -6 \] Row 2
\[ x_1 + 2x_2 + x_5 = 3 \] Row 3
\[ x_1, x_2, x_3, x_4, x_5 \geq 0 \] Non-negativity constraints

Steps for Dual Simplex

1. First convert all the constraints to equality form by introducing slack or surplus variables, as may be necessary.
2. Identify and start from a basic infeasible solution, satisfying optimality condition (all coefficients in the objective function row being non-negative). The optimality condition ensures that the solution remains always so in the subsequent iterations. The iterations force basic solutions toward the feasible space.

The starting solution is identified (as in the primal LP problem) and the simplex table for the starting solution constructed as in Table 2.3. In the starting solution for the example problem above, there are three basic variables: \( x_3 = -3 \), \( x_4 = -6 \), \( x_5 = 3 \), and two nonbasic variables \( x_1 \) and \( x_2 \) (Table 2.3).

Table 2.3 Starting Solution

<table>
<thead>
<tr>
<th>Basis</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>-3</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-3</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>-4</td>
<td>-3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-6</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>( z' )</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Ratio 2/abs(-4) = 1/abs(-3)
\( = 1/2 \) (tie), \( = 1/3 \) (min.)
\( x_1 \) enters and \( x_3 \) leaves

Iteration 1

<table>
<thead>
<tr>
<th>Basis</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>-5/3</td>
<td>0</td>
<td>1</td>
<td>-1/3</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>4/3</td>
<td>1</td>
<td>0</td>
<td>-1/3</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>-5/3</td>
<td>0</td>
<td>0</td>
<td>2/3</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( z' )</td>
<td>2/3</td>
<td>0</td>
<td>0</td>
<td>1/3</td>
<td>0</td>
<td>-2</td>
</tr>
</tbody>
</table>

Entering variable \( x_2 \) enters

Ratio \( 2/3 + 5/3 = 1/3 + 1/3 \)
\( = 2/5 \) (tie), \( = 1 \) (min.)
\( x_1 \) enters, \( x_2 \) leaves

Let \( x_1 \) leave
There is a tie for the departing variable. Break the tie arbitrarily. Let $x_3$ depart the basis.

**Iteration 2**

<table>
<thead>
<tr>
<th>Basis</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/5</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2/5</td>
<td>1/5</td>
<td>0</td>
</tr>
</tbody>
</table>

Optimum value of the objective function $z' = -12/5$. Thus $z = -z' = 12/5$. $x_1 = 3/5$, $x_2 = x_4 = x_5 = 0$.

3. The departing basic variable is identified first as one with the most negative value. This is variable $x_4$ in the starting solution for the example problem (Table 2.3: $x_4 = -6$). If all the basic variables are non-negative, the iterations end, and the feasible, optimal solution is reached.

4. The entering variable is selected next from among the nonbasic variables. For each nonbasic variable, determine the ratio of the coefficient in the objective function row to the absolute value of the corresponding coefficient in the row of the departing variable. The ratios need to be calculated only for those nonbasic variables with negative coefficients in the row of the departing variable. The nonbasic variable, which yields the least ratio, is the entering variable.

Note that in the primal simplex method, the entering variable is identified first and then the departing variable. On the other hand, in the dual simplex method, the departing variable is identified first and then the entering variable. The ratio test identifies the departing variable in the primal simplex, whereas in the dual simplex method, the ratio test (although administered in a different way) identifies the entering variable.

5. A tie for a departing variable, if it occurs, may be broken arbitrarily, like in the iteration 1 (Table 2.3) for the example problem above.

6. The row operations are performed (as in the primal simplex method) and iterations are continued till a feasible solution is arrived at.

7. A feasible solution is identified by the presence of all non-negative right-hand sides in the simplex table (except for the RHS of the objective function row, which may be negative, positive or zero). The earliest iteration, which achieves this condition, gives the feasible and therefore the optimal solution for the dual.

In the example problem above, feasibility is reached in iteration 2 with $x_1 = 3/5$, $x_2 = 6/5$, $x_3 = 0$, and the objective value $z' = -12/5$, or $z = -z' = 12/5$, i.e. the minimum value of $z$, the objective in the original problem, is 12/5.

**Choice of the Method of Solution**

A given linear programming problem of any form can be solved by either of the two simplex methods (primal simplex or dual simplex method), depending
on the feasibility of the initial basic solution. Depending on the actual form of the given LP problem, the constraints are first transformed to the equality form (by introducing slack or surplus variables, and no artificial variables) and the initial basic solution or the starting solution (feasible or infeasible) is identified. If the starting solution is a basic feasible solution, the primal simplex method is applied, or if the starting solution is a basic infeasible solution, the dual simplex method is applied.

**Identification of the Primal Solution from the Dual Solution** Sometimes, it is easier to form the dual and then solve the dual rather than the primal itself, especially if the primal has a large number of constraints. It is the number of constraints that increases computational time rather than the number of variables in LP, because of the number of rows to be transformed in the iterations.

It is possible to infer the values of the dual variables from the final simplex table of the primal problem, and vice versa. For example, consider the dual of the original example primal problem (Example 2.2.1) discussed. The dual is (setting \( z' = z' \)):

Minimize \( z' = 300y_1 + 200y_2 \)

subject to

\[
\begin{align*}
3y_1 + y_2 &\geq 2 \\
y_1 + y_2 &\geq 1 \\
y_1, y_2 &\geq 0
\end{align*}
\]

**Solution of the Dual** Writing the dual in the standard form with equality constraints,

Maximize \( (-z') = -300y_1 - 200y_2 \)

or

\[
\begin{align*}
(-z') + 300y_1 + 200y_2 &= 0 \\
3y_1 + y_2 - y_1 &= 2 \\
y_1 + y_2 - y_1 &= 1 \\
y_1, y_2, y_3, y_4 &\geq 0
\end{align*}
\]

where \( y_3 \) and \( y_4 \) are surplus variables.

Writing the problem in a way to facilitate a starting basic infeasible solution for dual simplex method,

\[
\begin{align*}
(-z') + 300y_1 + 200y_2 &= 0 \\
-3y_1 - y_2 + y_1 &= -2 \\
y_1 - y_2 + y_4 &= -1 \\
y_1, y_2, y_3, y_4 &\geq 0
\end{align*}
\]

Starting solution

<table>
<thead>
<tr>
<th>Basis</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
<th>( y_3 )</th>
<th>( y_4 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_1 )</td>
<td>-3</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( y_4 )</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

\( (-z') \)

<table>
<thead>
<tr>
<th>|</th>
<th>|</th>
<th>|</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>300</td>
<td>200</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\text{Ratio} \quad 300/3 &= 100 \\
200/1 &= 200
\end{align*}
\]
### Basics of Systems Techniques

#### Iteration 1

<table>
<thead>
<tr>
<th>Basis</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
<th>$y_4$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>1</td>
<td>1/3</td>
<td>-1/3</td>
<td>0</td>
<td>2/3</td>
</tr>
<tr>
<td>$y_4$</td>
<td>0</td>
<td>-2/3</td>
<td>-1/3</td>
<td>1</td>
<td>-1/3</td>
</tr>
<tr>
<td>($-z'$)</td>
<td>0</td>
<td>100</td>
<td>100</td>
<td>0</td>
<td>-200</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Ratio</th>
<th>100/2/3</th>
<th>100 + 1/3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>= 150</td>
<td>= 300</td>
</tr>
</tbody>
</table>

#### Iteration 2

<table>
<thead>
<tr>
<th>Basis</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
<th>$y_4$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>1</td>
<td>0</td>
<td>-1/2</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>$y_2$</td>
<td>0</td>
<td>1</td>
<td>1/2</td>
<td>-3/2</td>
<td>1/2</td>
</tr>
<tr>
<td>($-z'$)</td>
<td>0</td>
<td>0</td>
<td>50</td>
<td>150</td>
<td>-250</td>
</tr>
</tbody>
</table>

**Solution:** $y_1 = 1/2$, $y_2 = 1/2$, ($-z'$) = -250, or $z' = 250$.

The following points may be noted, referring to the final solution table.

1. The objective function value is the same in both the primal and dual solutions.
   - Primal: Optimum value of $z = 250$
   - Dual: Optimum value of $z' = 250$.
   - Thus $z_{\text{opt}} = z'_{\text{opt}} = 250$.

2. Note that the dual variables from the optimal solution are $y_1 = 1/2$ and $y_2 = 1/2$. The coefficient in the objective function row (in the dual) of the nonbasic variable $y_3$ is 50. Recall that $y_3$ is the slack variable in the dual constraint corresponding to the primal variable $x_1$. The optimal value of $x_1$ in the primal can be identified by the coefficient of the slack variable $y_3$ in the corresponding dual constraint, which is equal to 50. Thus $x_1 = 50$.
   - Similarly, the coefficient in the objective row for $x_4$, which is 150, will be equal to the optimal value of the primal variable $x_2$, which is associated with the dual constraint containing $y_4$ is 150. Thus, $x_2 = 150$.
   - Thus the primal solution: $x_1 = 50$, $x_2 = 150$ and $z = 250$ can be identified from the solution of the dual.

3. In a similar way, the dual solution can be inferred from the final simplex table (optimal solution) of the primal problem, Table 2.1.
   - Note that the coefficient of $x_1$ in the objective row of the final simplex table of the primal solution is 1/2, which is the same as the value of the dual variable, $y_3$, corresponding to the primal constraint in which $x_1$ is a slack variable. Similarly, the coefficient of $x_4$ in the objective row of the final simplex table of the primal solution is 1/2, which is the same as the value of the dual variable $y_4$, corresponding to the primal constraint in which $x_4$ is the slack variable. The optimum value of the objective function is the same in both.
4. In any iteration, the objective function value, $z$, in the primal solution, is less than or equal to the objective function value of the dual, i.e. $z \leq z'$. The optimum values of both are exactly the same, i.e. $z_{opt} = z'_{opt}$.

Because of these interrelationships between a primal and its dual, it is easier, as mentioned earlier, to solve the dual instead of the primal when the primal has more constraints than the number of decision variables.

As an illustration, let us consider the following primal problem with two decision variables and four constraints:

**Primal:**

Maximize $z = 2x_1 + x_2$

subject to

$3x_1 + x_2 < 300$

$x_1 + x_2 < 200$

$2x_1 + 5x_2 < 900$

$5x_1 + 2x_2 < 600$

$x_1, x_2 > 0$

The dual of this primal is

**Dual:**

Minimize $z' = 300y_1 + 200y_2 + 900y_3 + 600y_4$

subject to

$3y_1 + y_2 + 2y_3 + 5y_4 \geq 2$

$y_1 + 2y_2 + 5y_3 + 2y_4 \geq 1$

$y_1, y_2, y_3, y_4 \geq 0$

which has four decision variables and two constraints. Because of the lesser number of constraints, it is easier to solve the dual rather than the primal using the simplex method. The solution for the problem is $y_1 = y_2 = 1/2$, $x_1 = 50$, $x_2 = 150$, and $z = z' = 250$.

### 2.2.6 Matrix Form

It is advantageous to express an LP in the matrix form to facilitate an understanding of simplex operations, as we shall see in this section.

The standard form of a linear programming problem with equality constraints with all non-negative variables and right-hand sides can be expressed in matrix form as follows:

Maximize $z = CX$

subject to $(A, I)X = b$

$X \geq 0$

where $I$ is $(m \times m)$ identify matrix, $X$ is a column vector and $C$, a row vector given by

$C = (c_1, c_2, \ldots, c_m)^T$

and $A$ is $(m \times n)$ matrix, $b$ is a column vector given by

$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

The elements of $b$ are non-negative in the primal simplex method.
Example 2.2.3 Consider the LP problem

Maximize \[ z = 4x_1 + 5x_2 \]
subject to \[
2x_1 + 3x_2 \leq 12 \\
4x_1 + 2x_2 \leq 16 \\
x_1 + x_2 \leq 8 \\
x_1, x_2 \geq 0
\]

The problem is written in the standard form first.

Maximize \[ z = 4x_1 + 5x_2 + 0.x_3 + 0.x_4 + 0.x_5 \]
subject to \[
2x_1 + 3x_2 + x_3 = 12 \\
4x_1 + 2x_2 + x_4 = 16 \\
x_1 + x_2 + x_5 = 8 \\
x_1, x_2, x_3, x_4, x_5 \geq 0
\]

The problem is expressed in the matrix form as

\[
\begin{align*}
\text{Maximize} & \quad z = \begin{bmatrix} 4 & 5 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \\
\text{subject to} & \quad \begin{bmatrix} 2 & 3 & 1 & 0 & 0 \\ 4 & 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 12 \\ 16 \\ 18 \end{bmatrix} \\
& \quad x_j \geq 0 \quad j = 1, 2, \ldots, 5.
\end{align*}
\]

Note that \( x_1, x_2 \) are decision variables, and \( x_3, x_4, x_5 \) are slack variables.

This problem is expressed in matrix notation as

\[
\begin{align*}
\text{Maximize} & \quad z = CX \\
\text{subject to} & \quad (A, I) X = b \\
& \quad X \geq 0 \\
& \quad X = [x_1, x_2, x_3, x_4, x_5]^T \\
& \quad C = [4, 5, 0, 0, 0] \\
& \quad b = \begin{bmatrix} 12 \\ 16 \\ 18 \end{bmatrix} \\
& \quad A = \begin{bmatrix} 2 & 3 \\ 4 & 2 \\ 1 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{align*}
\]

The Simplex Table in Matrix Form

The general procedure in the simplex method is to use the basis \( B \) comprising slack variables \( x_{n+1}, x_{n+2}, \ldots, x_{n+m} \) as the starting basis. Thus the basis matrix
Systems Techniques in Water Resources

(containing coefficients of slack variables in the constraints as elements) is the identity matrix \( I \). A simplex iteration results in moving the solution to an adjacent corner point towards optimality. This is done everytime by exchanging one vector in \( B \) with a current nonbasic vector that will move the solution towards optimality.

Let the vector \( X \) be partitioned into \( X_I \) and \( X_{II} \), where \( X_I \) corresponds to the elements of \( X \) associated with the starting basis \( B = I \). For the LP in the standard form, \( x_{n+1}, x_{n+2}, \ldots, x_{n+m} \) are elements of the starting basis \( B = I \). The remaining elements, \( x_1, x_2, \ldots, x_n \) form the elements of \( X_I \). The vector \( C \) is partitioned into \( C_I \) and \( C_{II} \) to correspond to \( X_I \) and \( X_{II} \). The standard form of LP may thus be expressed as

\[
\begin{pmatrix}
1 & -C_I \\
0 & A & -C_{II} & I
\end{pmatrix}
\begin{pmatrix}
z \\
x_I \\
x_{II}
\end{pmatrix} =
\begin{pmatrix}
0 \\
\mathbf{b}
\end{pmatrix}
\]

At any iteration, let

- \( B \) be the basis of the solution,
- \( X_B \) be the vector of current basic variables,
- \( C_B \) be the vector (containing elements of \( C \)) associated with \( X_B \).

Thus the objective

\[ z = C_B \cdot X_B \]

and

\[ BX_B = \mathbf{b} \]

Going through matrix manipulations, a general simplex iteration can be expressed in the following matrix form:

<table>
<thead>
<tr>
<th>Basis</th>
<th>( \bar{X}_I )</th>
<th>( \bar{X}_{II} )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{X}_I )</td>
<td>( B^{-1}A )</td>
<td>( B^{-1} )</td>
<td>( B^{-1} \mathbf{b} )</td>
</tr>
<tr>
<td>( \bar{C}_I )</td>
<td>( B^{-1}A - C_I )</td>
<td>( B^{-1}C_{II} )</td>
<td>( C_{II}B^{-1} \mathbf{b} )</td>
</tr>
</tbody>
</table>

where \( B^{-1} \) is the inverse of the basis matrix \( B \). This representation is valid for any iteration with a given or chosen \( X_B \) and the associated \( B \). Note that \( C_I \) and \( C_{II} \) are vectors arising out of the starting solution (initial basic feasible solution) corresponding to \( X_I \) and \( X_{II} \), respectively, in the simplex method.

One striking feature of the matrix representation is that, once the basis \( B \) and the associated \( X_B \) are specified, all the other elements in the table are a function of \( B^{-1} \) and the data of the original LP problem. This feature offers a powerful tool to construct the entire simplex table for any iteration with an assumed or given basis.

For example, for the starting solution of the primal simplex method, \( X_B = X_{II} \), \( C_B = C_{II} = 0 \), \( B = I \) (\( \equiv B^{-1} \)) so that the starting table is arrived at as follows:

<table>
<thead>
<tr>
<th>Basis</th>
<th>( \bar{X}_I )</th>
<th>( \bar{X}_{II} )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{X}_I )</td>
<td>( A )</td>
<td>( I )</td>
<td>( \mathbf{b} )</td>
</tr>
<tr>
<td>( \bar{z} )</td>
<td>(-C_I )</td>
<td>( O )</td>
<td>( O )</td>
</tr>
</tbody>
</table>
The construction of the simplex table for an intermediate iteration is illustrated in the following example.

**Example 2.2.4** Consider the problem in Example 2.2.3. We shall construct the simplex table for a given basis and solve the problem using the simplex method.

Maximize \( z = 4x_1 + 5x_2 \)
subject to
\[
\begin{align*}
2x_1 + 3x_2 & \leq 12 \\
x_1 + x_2 & \leq 8 \\
x_1, x_2 & \geq 0
\end{align*}
\]

First write the problem in the standard form with equality constraints,
Maximize \( z = 4x_1 + 5x_2 + 0.0x_3 + 0.0x_4 + 0.0x_5 \)
subject to
\[
\begin{align*}
2x_1 + 3x_2 + 1.x_3 + 0.x_4 + 0.x_5 & = 12 \\
4x_1 + 2x_2 + 0.x_3 + 1.x_4 + 0.x_5 & = 16 \\
x_1 + x_2 + 0.x_3 + 0.x_4 + 1.x_5 & = 8 \\
x_1, x_2, x_3, x_4, x_5 & \geq 0.
\end{align*}
\]

We shall construct the simplex table for the basis \( X_B = (x_2, x_4, x_5) \). There can be only 3 basic variables for the problem and let them be \( x_2, x_4 \) and \( x_5 \). The remaining variables \( x_1 \) and \( x_3 \) are zeros, being non basic.

The basis \( B = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \), the column elements of \( B \) are obtained by the coefficients of the basic variables in the equality constraints.

\( C_B = \) vector of coefficients in the objective function associated with the basic variables
\( = (5, 0, 0) \)
\( C_2 = (4, 5) \) corresponding to \( X_I = [x_1, x_2]^T \)
\( C_{II} = (0, 0, 0) \) corresponding to \( X_{II} = (x_3, x_4, x_5)^T \)
\( A = \begin{bmatrix} 2 & 3 \\ 4 & 2 \\ 1 & 1 \end{bmatrix} \)

contains column vectors corresponding to the nonbasic variables in the starting solution
(Not that \( C_B, C_{II}, X_I, X_{II} \) and \( A \) will be the same for all the iteration of a given problem.)

For the given \( B \), \( B^{-1} = \begin{bmatrix} 1/3 & 0 & 0 \\ -2/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} \)
\( b = \begin{bmatrix} 12 \\ 16 \\ 8 \end{bmatrix} \)
The following may now be computed as

\[
B^{-1} A = \begin{bmatrix}
2/3 & 1 \\
8/3 & 0 \\
1/3 & 0
\end{bmatrix}
\]

\[C_B B^{-1} = (5/3, 0, 0)\]

\[C_B B^{-1} A = (10/3, 5)\]

\[C_B B^{-1} A - C_I = (5/3, 0, 0) - (0, 0, 0) = (5/3, 0, 0)\]

\[C_B B^{-1} b = \begin{bmatrix}
12 \\
6 \\
8
\end{bmatrix} = 20\]

\[B^{-1} b = \begin{bmatrix}
4 \\
4 \\
4
\end{bmatrix}\]

Substituting for the values in the matrix form of the table, we get

<table>
<thead>
<tr>
<th>Basis</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_2)</td>
<td>2/3</td>
<td>1</td>
<td>1/3</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>(x_4)</td>
<td>8/3</td>
<td>0</td>
<td>-2/3</td>
<td>1</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>(x_5)</td>
<td>1/3</td>
<td>0</td>
<td>-1/3</td>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>(z)</td>
<td>-2/3</td>
<td>0</td>
<td>5/3</td>
<td>0</td>
<td>0</td>
<td>20</td>
</tr>
</tbody>
</table>

The solution when the basis, \(X_B = (x_2, x_4, x_5)\) is \(x_2 = 4, x_4 = 8, x_5 = 4\) and \(z = 20\).

It is seen that the solution is not optimal because of the negative coefficient for the nonbasic variable, \(x_1\), in the objective row in the simplex table (-2/3).

Therefore an iteration is required to make the solution move towards optimality.

The optimal basis for this problem is \(X_B = (x_2, x_1, x_5)\)

\[B = \begin{bmatrix}
3 & 2 & 0 \\
2 & 4 & 0 \\
1 & 1 & 1
\end{bmatrix} \quad B^{-1} = \begin{bmatrix}
1/2 & -1/4 & 0 \\
-1/4 & 3/8 & 0 \\
-1/4 & -1/8 & 1
\end{bmatrix}\]

and the optimal solution is given by the final simplex table as follows:

<table>
<thead>
<tr>
<th>Basis</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_2)</td>
<td>0</td>
<td>1</td>
<td>1/2</td>
<td>-1/4</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(x_1)</td>
<td>1</td>
<td>0</td>
<td>-1/4</td>
<td>3/8</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>(x_5)</td>
<td>0</td>
<td>0</td>
<td>-1/4</td>
<td>-1/8</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>(z)</td>
<td>0</td>
<td>0</td>
<td>3/2</td>
<td>1/4</td>
<td>0</td>
<td>22</td>
</tr>
</tbody>
</table>

Note: As long as the same basic variables are used, the order of the rows does not matter in the table. For example, if \(X_B\) is chosen as \([x_3, x_1, x_5]^T\) instead of \((x_2, x_1, x_5)^T\) as in the above simplex table, the rows for the basic variables \(x_1\) and \(x_3\) will be interchanged (because \(B, B^{-1}\) will be different). What is
Important is $X_B$, $B$, $B^{-1}$ should be perfectly consistent with each other and correspond to the same basis.

It must be noted that the matrix form can be used to construct the simplex table for a given basis, but still iterations may be required to determine the optimal solution. The solution will be optimal only if the basis is optimal.

**Dual Variables in the Matrix Form**

The elements in the objective function row under the nonbasic variables (i.e., coefficients of the nonbasic variables in the objective row) are given by $C_B B^{-1}$, $-C_B$ under $X_B$. But recall that $C_B$ is the vector of the coefficients of basic variables in the objective function, $z = C_I X_I + C_B X_B$ in the starting solution, which means that $C_B = 0$. As mentioned earlier (Section 2.2.5), the coefficients under the nonbasic variables in the objective row in the optimal primal simplex table are the dual variables. The dual variables are given by

$$Y = C_B B^{-1},$$

where $B$ is the optimal basis of the primal simplex problem, $X_B = (x_2, x_3, x_5)^T$. For the example problem given above, $B = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 4 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$$B^{-1} = \begin{bmatrix} 1/2 & -1/4 & 0 \\ -1/4 & 3/8 & 0 \\ -1/4 & -1/8 & 1 \end{bmatrix}$$

$C_B = (5, 4, 0)$, $Y = C_B B^{-1} = (3/2, 1/4, 0)$

$\therefore$ The dual variables associated with the first, second, and third equality constraints are $y_1 = 3/2$, $y_2 = 1/4$ and $y_3 = 0$, respectively.

**LP Problem Viewed as a Resource Optimization Problem**

We may write a typical linear programming problem as

Maximize $z = \sum_{j=1}^{n} c_j x_j$

subject to $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$, $i = 1, 2, \ldots, m$

$x_j \leq 0$, $j = 1, 2, \ldots, n$

This is a problem with $n$ variables and $m$ constraints.

This problem may be looked at as a resource optimization problem: Consider a production process in which there are $n$ activities (inputs), $j = 1, 2, \ldots, n$. The level (the number of units) of activity $j$ is $x_j$. Let $c_j$ be the marginal profit associated with unit activity $j$. Then the total profit from all activities, $z = \sum_{j=1}^{n} c_j x_j$. 
There are constraints, however, on the activity levels. Let us say, there are $m$ resources (budget, material, manpower, etc.) which limit the levels of the $n$ activities, depending on the consumption of each resource for each unit of each activity. The index for resource is $i$ ($i = 1, 2, \ldots, m$). The level of resource $i$ is $b_i$. The resources are $b_1, b_2, \ldots, b_m$.

Let the allocation of resource $i$ to each unit of activity $x_j$ be $a_{ij}$.

Then resource “consumed” by $x_j$ is $a_{ij}x_j$.

The resource, $i$, consumed by all the $j$ activities is $\sum_{j=1}^{n} a_{ij}x_j$.

Since the resource, $i$, is limited to $b_i$, the constraint

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i$$

limits the usage of the resource $i$ to its availability. There are $m$ such constraints, one for each resource. The non-negativity constraints for $x_j$ imply that the activity levels cannot physically be negative. Thus the following summarizes the notation for a resource optimization problem.

$\begin{align*}
  & \text{$j$ = activity} \\
  & \text{$x_j$ = level of activity, non-negative} \\
  & \text{$c_j$ = profit from each unit of activity $x_j$} \\
  & \text{$i$ = resource} \\
  & \text{$b_i$ = level of resource $i$} \\
  & \text{$a_{ij}$ = units of resource $i$ allocated to or consumed by each unit of activity, $x_j$.}
\end{align*}$

**Dual Variables** Dual variables carry an important interpretation in terms of the LP being interpreted as a resource optimization problem. Recall the optimal simplex table in the example problem given earlier. The dual variables, as identified by the coefficients of the nonbasic variable in the objective row of the final table, are

$$\begin{align*}
  y_1 &= 3/2 \text{ corresponding to the constraint } 2x_1 + 3x_2 + x_3 = 12 \\
  y_2 &= 1/4 \text{ corresponding to the constraint } 4x_1 + 2x_2 + x_4 = 16 \\
  y_3 &= 0 \text{ corresponding to the constraint } x_2 + x_5 = 8
\end{align*}$$

The objective row equation for the optimal solution is

$$z = 0.5x_1 + 0.5x_2 + 3/2x_3 + 1/4x_4 + 0.5x_5 = 22$$

In the optimal solution, $x_1 = 3$, $x_2 = 2$ and $x_3 = 3$, and $x_4 = x_5 = 0$.

The following observations are made:

A change in the value of the nonbasic variables, $x_3$ and $x_4$, (which are currently zero in the optimal solution), is bound to change the right-hand side values and the solution including the value of the objective function. If $x_3$ increases (from zero) by one unit, the right side changes from 12 to 13 and the objective value by 3/2, as inferred from the objective row of the final table. Thus, a unit change in the value of the constraint in which $x_3$ appears changes the value of the objective by $y_1$ (the dual variable for the constraint associated with the slack variable $x_3$ in the primal problem). This is valid only as long as $x_j$ varies in such a way that the optimal basis $(x_2, x_3, x_4)$ does not change (if $x_j$
Basics of Systems Techniques

is changed indefinitely then one or other of the variables \( x_1 \) and \( x_2 \) will be forced out of the basis, as no variable can be less than zero.

Thus the dual variable corresponding to a primal constraint represents the change in the value of the objective function for a given change in the value of the constraint—in other words, the worth of unit resource represented by the constraint.

It may be observed that in the third constraint, the dual variable is zero (under the variable \( x_5 \)), implying that a change in \( x_5 \) does not change the objective function value, as the coefficient \( x_5 \) in the objective is zero. This indicates that the original constraint \( x_1 + x_2 \leq 8 \) is redundant for the optimal solution (\( x_1 = 3, x_2 = 2 \)). If the basis should change because of a change in the value of the resources (right-hand side values), the solution itself changes and so would the dual variables.

The dual variables are referred to as dual prices or shadow prices in economic analysis.

2.2.7 Sensitivity Analysis

Sensitivity analysis is an exercise of obtaining the new solution corresponding to a change in the data of the original LP problem, given the original problem and the final simplex table, without solving afresh the new problem with changed data. The starting point is the final simplex table of the original problem, followed by an investigation of the extent to which the current optimal solution will be affected because of a change in the data of the original problem. The new solution is determined by further iterations. In many cases, the number of these iterations will be far less than those required to solve the problem afresh from its new starting solution.

For discussion in this section, changes in the original data of \( C_I, C_{II}, \) and \( b \) (in the matrix notation) alone will be considered. Changes in the matrix \( A \), addition or deletion of a constraint, addition or deletion of a variable, which are also covered under sensitivity analysis in textbooks on linear programming, are not covered here as these are not as common as the changes in \( C \) and \( b \) in engineering problems.

A change in the data of the original problem may affect optimality or feasibility, or both optimality and feasibility, of the current solution. Three cases are possible.

1. Changes that can affect only optimality—this can occur because of a change in the coefficients in the objective function \( C_I, C_{II} \).
2. Changes that can affect only feasibility—this can occur because of a change in the right-hand side values, \( b \).
3. Changes that can affect both optimality and feasibility—this can occur because of a simultaneous change in \( (C_I, C_{II}) \) and \( b \).

The general procedure to determine the new solution in these three cases are:

1. In the first case, use simplex method to the new table and continue iterations till optimality is reached.
2. In the second case, apply dual simplex method to the new table till feasibility is restored.

3. In the third case, first apply the primal simplex method to the new table disregarding infeasibility till optimality is reached, and then apply dual simplex method to recover feasibility, in that order.

**Example 2.2.4** Example 2.2.4 will be used to illustrate the changes mentioned in the three cases:

The original problem written in the standard form is

Maximize \( z = 4x_1 + 5x_2 \)

subject to

\[
\begin{align*}
2x_1 + 3x_2 + x_3 &= 12 \\
4x_1 + 2x_2 + x_4 &= 16 \\
x_1 + x_2 + x_5 &= 8 \\
x_1, x_2, x_3, x_4, x_5 &\geq 0
\end{align*}
\]

where \( x_3, x_4, x_5 \) are slack variables. The final simplex table is

<table>
<thead>
<tr>
<th>Basis</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>1</td>
<td>( 1/2 )</td>
<td>(-1/4)</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>1</td>
<td>0</td>
<td>(-1/4)</td>
<td>(3/8)</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>( z )</td>
<td>0</td>
<td>0</td>
<td>(3/2)</td>
<td>(1/4)</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Case I. Changes that can affect optimality

Let the new objective function be

Maximize \( z = 3x_1 + 8x_2 \)

Note that with this new objective function, \( C_B, C_I \) will change, while \( A, B^{-1} \) remain the same. Because of this, only the objective row elements will change in the final table. These changes are best tracked using the matrix form of the simplex table.

Now, in the changed problem \( C_B = (8, 3, 0) \) and \( C_I = (3, 8) \)

\[
X_B = (x_2, x_1, x_5)^T, \quad A = \begin{bmatrix} 2 & 3 \\ 4 & 2 \\ 1 & 1 \end{bmatrix} \text{ as before;}
\]

\[
C_B = (3, 8), \quad C_I = (0, 0, 0), \quad b = \begin{bmatrix} 12 \\ 16 \end{bmatrix}
\]

\[
C_B B^{-1} = \begin{bmatrix} 8 & 3 & 0 \\ -1/4 & 3/8 & 0 \end{bmatrix} = (13/4, -7/8, 0)
\]

\[
C_B B^{-1} A = (13/4, -7/8, 0) \begin{bmatrix} 2 & 3 \\ 4 & 2 \\ 1 & 1 \end{bmatrix} = (3, 8)
\]
The table will change as follows:

<table>
<thead>
<tr>
<th>Basis</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>1/2</td>
<td>-1/4</td>
<td>0</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>-1/4</td>
<td>3/8</td>
<td>0</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>0</td>
<td>-1/4</td>
<td>-1/8</td>
<td>1</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>0</td>
<td>0</td>
<td>13/4</td>
<td>-7/8</td>
<td>0</td>
</tr>
</tbody>
</table>

This is nonoptimal because of the negative value $(-7/8)$ under $x_4$ in the $z$-row.

Another iteration using the primal simplex method gives the optimal solution to the new problem. Try it.

<table>
<thead>
<tr>
<th>Basis</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2$</td>
<td>2/3</td>
<td>1</td>
<td>1/3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_4$</td>
<td>8/3</td>
<td>0</td>
<td>-2/3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_3$</td>
<td>2/3</td>
<td>0</td>
<td>-1/3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>7/12</td>
<td>0</td>
<td>8/3</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Thus the new optimal solution is $x_1 = 0$, $x_2 = 4$ and $z = 32$.

**Case II. Changes that Can Affect Feasibility**

1. Let the new right-hand values in the changed problem be

   \[
   \text{new } b = \begin{bmatrix} 13 \\ 16 \\ 8 \end{bmatrix}, \text{ i.e. the RHS of the first constraint increased by 1.}
   \]

   A change in $b$ will cause a change only in the RHS in the simplex table.

   New $X_B = B^{-1}b$.

   \[
   \begin{bmatrix}
   1/2 & -1/4 & 0 & 13 & 5/2 \\
   -1/4 & 3/8 & 0 & 16 & 11/4 \\
   -1/4 & -1/8 & 1 & 8 & 11/4
   \end{bmatrix}
   \]

   This solution is still feasible as the RHS are non-negative in each constraint.

   $x_2 = 5/2$, $x_1 = 11/4$ and $x_3 = 11/4$.

   New objective value $z = C_B B^{-1}b$

   \[
   C_B B^{-1} b = (3/2, 14, 0)
   \]

   \[
   \therefore \quad C_B B^{-1} b = (3/2, 14, 0) \begin{bmatrix} 13 \\ 16 \\ 8 \end{bmatrix} = \frac{47}{2}
   \]
Change in objective value = \( \frac{47}{2} - 22 = 3/2 = y_1 \).

2. Let the new RHS values be \( b = \begin{bmatrix} 24 \\ 24 \\ 8 \end{bmatrix} \).

Then \( \text{new } X_B = B^{-1} b = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{3}{8} & 0 \\ -\frac{1}{4} & -\frac{1}{8} & 1 \end{bmatrix} \begin{bmatrix} 24 \\ 24 \\ 8 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -1 \end{bmatrix} \).

This solution is \( x_2 = 6, \ x_1 = 3, \ x_3 = -1 \) (negative) and is infeasible as \( x_5 \) is negative.

The new simplex table for the basis \((x_2, \ x_1, \ x_5)^T\) is as follows:

<table>
<thead>
<tr>
<th>Basis</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>1</td>
<td>\frac{1}{2}</td>
<td>-\frac{1}{4}</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>1</td>
<td>0</td>
<td>-\frac{1}{4}</td>
<td>\frac{3}{8}</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>( x_5 )</td>
<td>0</td>
<td>0</td>
<td>-\frac{1}{4}</td>
<td>-\frac{1}{8}</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( z )</td>
<td>0</td>
<td>0</td>
<td>\frac{3}{2}</td>
<td>\frac{1}{4}</td>
<td>0</td>
<td>42</td>
</tr>
</tbody>
</table>

With this infeasible solution as the starting solution, apply dual simplex method. \( x_5 \) goes out of the basis.

Ratio for \( x_3 \): \( (3/2) / \text{abs} \left( -\frac{1}{4} \right) = 6 \)

Ratio for \( x_4 \): \( (1/4) / \text{abs} \left( -\frac{1}{8} \right) = 2 \) (min)

\( x_4 \) enters and \( x_5 \) goes out. Row transformations yield the simplex table for the new basis \([x_2, \ x_1, \ x_4]^T\) as follows. Try it.

<table>
<thead>
<tr>
<th>Basis</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>0</td>
<td>0</td>
<td>+2</td>
<td>1</td>
<td>-8</td>
</tr>
<tr>
<td>( z )</td>
<td>0</td>
<td>0</td>
<td>\frac{3}{2}</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The new solution is \( x_2 = 8, \ x_1 = 0, \ x_4 = 8, \ z = 20 \) (optimal).

Check: For the new basis \( C_B = (5, 4, 0) \), \( B = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 0 \end{bmatrix} \)

\( B^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 0 & 3 \\ 2 & 1 & -8 \end{bmatrix} \)

It can be verified that \( Y = C_B \ B^{-1} = (1/2, \ 0, \ 1) \)

and \( z = C_B \ B^{-1} b = 20 \).
2.2.8 Piecewise Linearization

In a linear programming problem, both the objective function and the constraints are linear. However, even if the objective function and/or the constraints in a problem are nonlinear, it may still be possible, under certain conditions, to use LP, if the nonlinear expressions can be expressed as piecewise linear segments. This would require additional variables and constraints to be introduced into the original problem.

This is an articulation process in the model formulation to suit the requirements of the LP. The following formulations of linear programming may be used to maximize a nonlinear concave function or to minimize a nonlinear convex function, if the functions can be made piecewise linear.

Maximization of a Concave Function

The nonlinear function, \( f(x) \), is expressed as a piecewise linear function consisting of segments, with the slope of the function in each reducing as \( x \) increases.

\[
\text{Method 1}
\]

The following is the model:

Maximize \( f(x) = w_1 f(a_1) + w_2 f(a_2) + w_3 f(a_3) + \ldots + w_n f(a_n) \)

subject to

\[
w_1 a_1 + w_2 a_2 + w_3 a_3 + \ldots + w_n a_n = x
\]

\[
w_1 + w_2 + w_3 + \ldots + w_n = 1
\]

In the optimal solution, no more than two of the weights, \( w_i \), will be positive, all others being zero. For example, if the value of \( x \) falls within the segment between \( a_j \) and \( a_{j+1} \), only \( w_j \) and \( w_{j+1} \) will be nonzero and all other \( w_i \) will be zero. The maximum \( f(x) \) is approximated by

\[
f(x) = w_j f(a_j) + w_{j+1} f(a_{j+1}) = w_j f(a_j) + [1 - w_j f(a_{j+1})]
\]

\[
\text{Method 2}
\]

Let the slopes of the linear segments be \( s_1, s_2, \ldots \), where \( s_1 > s_2 > s_3 \ldots \)

Then the problem is to

Maximize \( f(x) = s_1 x_1 + s_2 x_2 + s_3 x_3 + \ldots = \sum s_i x_i \)

subject to

\[
a_1 x_1 + a_2 x_2 + \ldots = x
\]

\[
x_i \leq a_{j+1} - a_j \text{ for all segments } j.
\]
An alternate method is to use integer variables to determine the function (see Example, Fig. 2.8). This method will be useful in dealing with nonlinear separable functions. The use of integer variables in LP model formulation of nonlinear functions will be discussed in the application of linear programming for water resources planning problems later in the book [Chapters 5, 7 and 9]. If it is required to minimize a convex function, the function with a negative sign before it (to make it concave) can be maximized in the same fashion as shown above.

**Use of LP to Solve Some Nonlinear Programming Problems**

1. Min–max Problem  
   
   **Objective:** Minimize \( \max \{X_i\} \)

   where \( X_i \) takes on different values.

   Define an upper bound for \( X_i \) (for all \( i \)), say \( K \), and minimize \( K \). The formulation is

   Minimize \( K \)
   
   subject to \( X_i \leq K \) for all \( i \).

2. Max–min Problem  
   
   **Objective:** Maximize \( \min \{X_i\} \)

   Define a lower bound, say \( K \) for \( X_i \) for all \( i \) and maximize \( K \). The formulation is

   Maximize \( K \)
   
   subject to \( X_i \geq K \) for all \( i \).

3. Minimize the Absolute Value of an Unrestricted Variable  
   
   Let the unrestricted variable be \( X \), the modulus of which must be minimized.

   **Objective:** Minimize \(|X|\)

   Express \( X \) as the difference between two non-negative variables, say \( U \) and \( V \), and minimize \((U + V)\). The formulation is

   Minimize \((U + V)\)
   
   subject to \( X = U - V \).

4. Minimize the Absolute Value of the Deviation between Two Non-negative Variables  
   
   Let the absolute value of deviation be equal to \( D \) (non-negative), and the two variables be \( X \) and \( Y \).

   **Objective** is to minimize \(|X - Y| = D\)

   Introduce two constraints \( X - Y \leq D \)
   
   and \( Y - X \leq D \)

   and minimize \( D \). The formulation is

   Minimize \( D \)
   
   subject to \( X - Y \leq D \)
   
   \( Y - X \leq D \)

   Or,

   let \( X - Y = L - M \), and minimize \((L + M)\).
This formulation will be useful in occasions when the deviation of a decision variable from an unknown target (another decision variable) is to be minimized.

**Formulation Using Integer Variables**

It is necessary sometimes to specify that certain constraints operate only under a specified condition. This specification cannot be met in general LP, where the variables can take on any value as long as they are non-negative. It is here that the introduction of integer variables in the formulation of LP is useful. Some simple problems are illustrated below.

5. **Fixed Cost Problem** To minimize the total costs comprising both fixed and variable costs.

Let the total cost of constructing a reservoir, $TC$, be

$$TC = FC + \alpha S \quad \text{for } S > 0$$

where $\alpha$ is the variable cost per unit of storage, and $S$ is the storage required in the reservoir.

Obviously when $S = 0$, the total cost $TC$ also is zero, as no reservoir need to be constructed. Thus

$$TC = 0 \quad \text{if } S = 0,$$

$$TC = FC + \alpha \cdot S \quad \text{if } S > 0.$$  

Introduce an integer variable, $\beta$, which can take on only two values 0 and 1. This is done by constraining the integer variable, $\beta \leq 1$.

The formulation is

Minimize $\beta(FC) + \alpha \cdot S$

subject to

$$S \leq \beta \cdot K$$

$$\beta \leq 1$$

where $K$ is an upper bound for the storage $S$. In the formulation $\beta$ tends to assume the lowest value, if possible. If $S > 0$, then $\beta$ is forced to be equal to 1, and the objective function adds the fixed cost $FC$ to the variable cost. Instead, if $\beta = 0$, then $S = 0$ and $TC = 0$. Thus both conditions are satisfied.

6. **Separable Functions: Continuous or Discontinuous** In this technique, each nonlinear function is approximated as a series of linear fixed-cost functions.
For a single segment

\[ F(X) = \beta \cdot F(0) + \alpha X \]

where \( \alpha \) is the slope of the linearized function, \( F(0) \) is the fixed cost for \( X > 0 \), \( \beta \) is a \((0,1)\) integer variable, and \( M \) is an arbitrarily large constant.

**Multiple Segments**

The nonlinear function is approximated by a number of piecewise linear segments. Each linear segment is expressed as a fixed-cost function. There is no requirement that the nonlinear function be either convex or concave. Still LP can be used to read the value of the function \( F(X) \) for a given value of \( X \). The problem may be one of maximization or minimization. An example is illustrated in Fig. 2.8. In a larger problem \( X \) itself could be a decision variable.

Given: \( a, b, c, d, F(0), F(a), F(b), F(c), F(d) \)

Introduce: \( x_1, x_2, x_3, x_4 \) – variables

\( \beta_1, \beta_2, \beta_3, \beta_4 \) – integer variables \((0,1)\)

Maximize or minimize

\[ F(X) = \{\beta_1 F(0) + f_1 x_1\} + \{\beta_2 F(a) + f_2 x_2\} + \{\beta_3 F(c) + f_3 x_3\} + \{\beta_4 F(d) + f_4 x_4\} \]
subject to \( X = (\beta_0 + x_1) + (\beta_1 b + x_2) + (\beta_2 c + x_3) \)
\( x_1 \leq \beta_0 (a - 0); x_1 \leq \beta_1 (b - a); x_1 \leq \beta_2 (c - b); x_4 \leq \beta_3 (d - c) \)
\( \beta_0 + \beta_1 + \beta_2 + \beta_3 \leq 1 \)
\( \beta_0, \beta_1, \beta_2, \beta_3 \) are integer variables
\( X \leq X_{\text{max}} \)

where \( X_{\text{max}} \) is an upper bound for \( X \) and a constant.

**Example 2.2.6**

The cost function of a reservoir of capacity \( K \) is as shown. Formulate a mixed integer LP model to determine the cost, when the reservoir capacity is 17 units. What is the solution? In this case \( X = 17 \).

![Graph of cost function and reservoir capacity](image)

Slope \( f_1 = 20/10 = 2 \); Slope \( f_2 = 16/4 = 4 \); Slope \( f_3 = 15/5 = 3 \)

Min \( C = (20\beta_1 + 2x_1) + (40\beta_2 + 4x_2) + (56\beta_3 + 3x_3) \)

subject to
\( 17 = (0\beta_1 + x_1) + (10\beta_2 + x_2) + (14\beta_3 + x_3) \)
\( x_1 \leq 10\beta_1 \)
\( x_2 \leq 4\beta_2 \)
\( x_3 \leq 5\beta_3 \)
\( \beta_0 + \beta_1 + \beta_2 + \beta_3 \leq 1 \)
\( \beta_0, \beta_1, \beta_2, \beta_3 \) are integers

The solution is \( \beta_0 = \beta_2 = 0, \beta_1 = 1, x_1 = x_2 = 0, x_3 = 3; C = 65 \).

**Problems**

Solve the following problems using simplex/dual simplex method.

2.2.1 Maximize \( z = 5x_1 + 8x_2 \)
subject to \( 2x_1 + 3x_2 \geq 15 \)
\( 3x_1 + 5x_2 \leq 60 \)
\( x_1 + x_2 = 18 \)
\( x_1, x_2 \geq 0 \)

(Ams: \( x_1 = 15; x_2 = 3 \))
2.2.2 Maximize \( z = 3x_1 + 4x_2 \)
subject to
\[
\begin{align*}
5x_1 + x_2 & \geq 45 \\
3x_1 + 5x_2 & \leq 72 \\
2x_1 + x_2 & \leq 24 \\
x_1, x_2 & \geq 0
\end{align*}
\]
(Ans: \( x_1 = 7; x_2 = 10; z = 61 \))

2.2.3 Maximize \( z = 2x_1 + 5x_2 \)
subject to
\[
\begin{align*}
3x_1 + x_2 & \geq 30 \\
5x_1 + 3x_2 & \geq 90 \\
x_1 + 2x_2 & \leq 15 \\
x_1, x_2 & \geq 0
\end{align*}
\]
(Ans: \( x_1 = 9; x_2 = 3; z = 33 \))

2.2.4 Minimize \( z = 3x_1 + 4x_2 \)
subject to
\[
\begin{align*}
x_1 + 3x_2 & \geq 15 \\
3x_1 + 5x_2 & \leq 72 \\
2x_1 + x_2 & = 24 \\
x_1, x_2 & \geq 0
\end{align*}
\]
(Ans: \( x_1 = 57/5; x_2 = 6/5; z = 39 \))

2.2.5 Solve by graphical method
(a) Maximize \( z = 5x_1 + 7x_2 \)
subject to
\[
\begin{align*}
3x_1 + 4x_2 & \leq 15 \\
2x_1 + 3x_2 & \geq 12 \\
x_1, x_2 & \geq 0
\end{align*}
\]
(b) Verify the solution by simplex method. Comment on the nature of the solution.
(Ans: \( x_1 = 0, x_2 = 4, z_{\text{max}} = 28 \))

2.2.6 Maximize \( z = 7x_1 + 5x_2 \)
subject to
\[
\begin{align*}
2x_1 + 3x_2 & \leq 12 \\
3x_1 + x_2 & \geq 6 \\
5x_1 + 3x_2 & \leq 15 \\
x_1, x_2 & \geq 0
\end{align*}
\]
(Ans: \( x_1 = 1.0, x_2 = 10/3, z_{\text{max}} = 71/3 \))

2.2.7 Maximize \( z = 3x_1 + 5x_2 + 2x_3 \)
subject to
\[
\begin{align*}
-2x_1 - x_2 & \leq -2 \\
x_1 + 4x_2 + 2x_3 & = 5 \\
x_1, x_2, x_3 & \geq 0
\end{align*}
\]
(Ans: \( x_1 = 1, x_2 = 1, x_3 = 0, z_{\text{max}} = 8 \))

2.2.8 Formulate a linear programming problem to maximize the total income and determine the areas \( x_1 \) and \( x_2 \) under crop 1 and crop 2, respectively, in hectares given the following data.

<table>
<thead>
<tr>
<th>Water Units/ha</th>
<th>Fertilizer Units/ha</th>
<th>Income/ha</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crop</td>
<td>Unit/ha required</td>
<td>Cost/Unit (Rs)</td>
</tr>
<tr>
<td>1</td>
<td>( w_1 )</td>
<td>( p_1 )</td>
</tr>
<tr>
<td>2</td>
<td>( w_2 )</td>
<td>( p_2 )</td>
</tr>
</tbody>
</table>
The following are resource limitations. 
Water availability is limited to \( W \) units. 
Fertilizer availability is limited to \( F \) units. 
Land availability is limited to \( A \) hectares. 
Money available for investment is limited to \( B \) (Rs).

2.2.9 Consider the LP problem.

Maximize \( z = -5x_1 + 5x_2 + 13x_3 \)
subject to
\[-x_1 + x_2 + 3x_3 \leq 20\]
\[12x_1 + 4x_2 + 10x_3 \leq 90\]
\[x_1, x_2, x_3 \geq 0\]

Let \( x_4 \) and \( x_5 \) be the slack variables for the first and second constraints respectively.

(a) Given that \([x_2, x_5]\) is the optimal basis in the solution, construct the complete final simplex table using the matrix approach.

(b) Find the range of RHS values (min and max) of the constraint in the original problem within which the optimal basis does not change.

(c) Determine the new optimal solution if the RHS values in the original problem change to 15 and 80 in place of 20 and 90, respectively.

(Ans: (b) min = 0, max = 22.5 (c) \( x_1 = 0, x_2 = 15, x_3 = 0 \))

2.2.10 (a) Use the simplex method to solve the following problem and verify the solution by the graphical method.

Minimize \( z = 3x_1 + 8x_2 \)
subject to
\[x_1 + 3x_2 \geq 12\]
\[3x_1 + 5x_2 \geq 30\]
\[x_1, x_2 \geq 0.\]

(Ans: \( x_1 = 15/2, x_2 = 3/2, z_{\text{min}} = 69/2 \))

(b) Identify the dual variables of the constraints of the problem in the final simplex table.

(Ans: \( y_1 = 9/4, y_2 = 1/4 \))

2.2.11 The total cost (fixed cost + variable cost) of constructing a reservoir is given as a function of the reservoir capacity, \( K \), as follows:

Fixed Cost: 20
Variable cost: 2 per unit capacity for \( 0 < K \leq 10 \);
4 per unit capacity for \( 10 < K \leq 14 \);
3 per unit capacity for \( 14 < K \leq 19 \).

Formulate a mixed integer linear programming problem to determine the total cost for a given capacity.

What should be the values \( x_1, x_2 \) and \( x_3 \) in the three segments of the axis representing capacity (on a plot of total cost vs capacity) for \( K = 17 \).

(Ans: \( x_1 = 10, x_2 = 4, x_3 = 3; \) Total cost = 65)
2.3 DYNAMIC PROGRAMMING

2.3.1 Introduction

Dynamic programming (DP) is ideally suited for sequential decision problems. Sequential (or multistage) decision problems are those in which decisions are made sequentially, one after another, based on the state of the system. An example of a sequential decision problem in water resources is the reservoir operation problem, in which release decisions need to be made sequentially across time periods (e.g. months) based on the available water in storage in a period. In this section, we discuss the formulation and solution of DP problems. Unlike linear programming problems, DP problems are not amenable to a single, standard algebraic formulation. Different types of sequential decision problems may need to be formulated differently considering specific features of the problems, and therefore, DP is said to be as much an art as it is a mathematical technique. In this book, we deal with only discrete DP, in which the variables are allowed to take only discrete values. The continuous state DP, which allows variables to take continuous values, has limited applications in water resources and therefore is not discussed here. The method of computations of discrete DP is illustrated with examples of water allocation, reservoir operation, capacity expansion and shortest route problems.

A single-stage decision problem may be represented as in Fig. 2.9. In this representation, $S$ is the input, $X$ is the decision and $T$ is the output. Because of decision $X$ for input $S$, there is a return $R$ which is, in general, a function of both $S$ and $X$. The transformation of input $S$ to output $T$ is called the state transformation. In a serial multistage decision problem, the output $T$ from one stage forms the input $S$ to the next stage, and a decision is associated with each stage.

A typical serial multistage decision problem, consisting of $n$ stages, may be represented as in Fig. 2.10 (e.g. Rao, 1999). $S_n$ is the input to stage $n$, $x_n$ is the decision taken at stage $n$, and $R_n$ is the return at stage $n$, corresponding to the decision $x_n$ for the input $S_n$. As a result of this decision, the input $S_n$ gets transformed into output $S_{n-1}$, that forms the input to the next stage $n-1$. As an example of a multistage decision problem, consider the problem of water allocation among $n$ users. Each user in such a problem defines one stage in the
multistage decision problem. The water allocated to a particular user, $i$, constitutes the decision $x_i$ at that stage. Starting with a known amount of available water $Q$, the input to the last stage $n$ (i.e. $S_n$) will be equal to $Q$. Based on the decision taken at the stage $n$ (i.e. the amount of water allocated to the user $n$), the amount of water available at stage $n-1$ will be $S_{n-1} = S_n - x_n$, which defines the state transformation. $S_{n-1}$ forms the input to stage $n-1$ and is equal to the amount of water available to be allocated to all the remaining stages, including the stage $n-1$. The return $R_n$ may be the monetary returns obtained from allocating an amount of water $x_n$ to user $n$. The allocation decision at any stage depends on the water available at that stage, after allocation to all previous stages.

The objective of a multistage decision problem as represented above, is to find the values of $x_1, x_2, x_3, \ldots, x_n$ to maximize a function of the returns (e.g. to maximize the sum of returns over all stages), while satisfying the state transformation equation. The input $S_i$ at any stage $i$ is called the state variable. The state variable defines the state of the system at a particular stage. In the water allocation problem mentioned earlier, the state variable at a particular stage is the water available for allocation to all the remaining users. It may be noted that a multistage problem may consist of more than one state variable.

### 2.3.2 Solution of DP problems

The solution procedure for DP problems is based on Bellman’s principle of optimality, which states, “An optimal policy (a set of decisions) has the property that whatever the initial state and initial decisions are, the remaining decisions must constitute an optimal policy with respect to the state resulting from the initial decision.” This principle implies that, given the state $S_i$ of the system at a stage $i$, one must proceed optimally till the last stage, irrespective of how one arrived at the state $S_i$. The solution procedure using this principle involves dividing the original problem with $n$ decision variables into $n$ subproblems, each with one decision variable. The subproblems are then solved recursively, one at each stage, either with a backward recursion or a forward recursion. The solution procedure is illustrated with a water allocation problem in the following paragraphs, by both backward and forward recurrences.

In serial multistage decision making problems, such as those discussed in this section, the stage numbers may be assigned in increasing order either in the forward direction or in the backward direction, no matter which direction of recursion we use, forward or backward. We must, however, define the state variable, state transformation equation and the recursive relationship at a given stage.
stage very precisely. The examples presented in this section illustrate different conventions that may be followed.

**Backward Recursion**

In backward recursion the computations proceed backwards, as illustrated in Fig. 2.11 for a three-user water allocation problem.

We define:

- $S_1$: Amount of water available for allocation at Stage 1 (to User 3);
- $S_2$: Amount of water available for allocation at Stage 2 (to Users 2 and 3 together);
- $S_3$: Amount of water available for allocation at Stage 3 (to Users 1, 2, and 3 together);

- $f_1(S_1)$: the maximum return due to allocation of $S_1$;
- $f_2(S_2)$: the maximum return due to allocation of $S_2$;
- $f_3(S_3)$: the maximum return due to allocation of $S_3$.

Only one user, User 3, is considered in the first stage, and optimal allocation to that user is obtained for all possible values of the state variable, $S_1$ (the amount
of water available for allocation at Stage 1). The problem is written, for Stage 1, as,

\[ f^*_1(S_1) = \max \{ R_1(x_1) \} \]

where \( f^*_1(S_1) \) is the maximum return at Stage 1 for given \( S_1 \), and \( R_1(x_1) \) is the return from allocating an amount of water \( x_1 \) to User 3. The bound for \( x_1 \) indicates that the search for the optimal value is carried out over all values that \( x_1 \) can take, viz. all values less than or equal to \( S_1 \), the water available for allocation at Stage 1. \( S_1 \) itself may take values less than or equal to \( Q \), the total amount of water available for allocation to all users.

In the second stage, two users are considered together, as shown in Fig. 2.11, at Stage 2. The state variable \( S_2 \) is the amount of water available to be allocated to User 2 and User 3 together. The problem is solved for all possible values of \( S_2 \), and the optimal value is obtained as

\[ f^*_2(S_2) = \max \{ R_2(x_2) + f^*_1(S_2 - x_2) \} \]

where \( f^*_2(S_2) \) is the maximum return at Stage 2 for given \( S_2 \) and \( R_2(x_2) \) is the return from allocating an amount of water \( x_2 \) to User 2. Note that the water available for allocation to User 3 (Stage 1) is \( S_2 - x_2 \), when an allocation of \( x_2 \) is made to User 2 out of the quantity, \( S_2 \), available for allocation at Stage 2. For this availability (\( S_2 - x_2 \)) for User 3, the maximum return is \( f^*_1(S_2 - x_2) \), which is obtained from the Stage 1 computations.

In the third (last) stage, all the three users are included in the optimization. The state variable, \( S_3 \), is the amount of water available for allocation to all the three users, which is \( Q \). The problem is solved for this value of \( S_3 \), and the optimal value is obtained as

\[ f^*_3(S_3) = \max \{ R_3(x_3) + f^*_2(S_3 - x_3) \} \]

where \( f^*_3(S_3) \) is the maximum return at Stage 3 for given \( S_3 \) and \( R_3(x_3) \) is the return from allocating an amount of water \( x_3 \) to User 1. When the problem is solved for the last stage, the entire amount, \( Q \), is optimally allocated among the three users. The individual allocations are then obtained by tracing back the solution. This general procedure may be adopted for any resource allocation problem that has \( n \) users. In backward recursion, the computations start with the last user, User \( n \), and proceed backwards stagewise, up to User 1, adding one user at every stage to the number of users included in the previous stage computations. At every stage in the backward recursion, we look forward to obtain optimal solution from the other stages that are already solved.

For discrete DP problems, the solution is conveniently obtained through a tabular procedure, as illustrated for a simple application of the water allocation problem.
Example 2.3.1 Water Allocation Problem

A total of 6 units of water is to be allocated optimally to three users. The allocation is made in discrete steps of one unit ranging from 0 to 6. With the three users denoted as User 1, User 2 and User 3 respectively, the returns obtained from the users for a given allocation are given in the following table.

<table>
<thead>
<tr>
<th>Amount of Water Allocated</th>
<th>Return from User 1</th>
<th>Return from User 2</th>
<th>Return from User 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>xR1</td>
<td>R1(x)</td>
<td>R2(x)</td>
<td>R3(x)</td>
</tr>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
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</tr>
<tr>
<td>2</td>
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<td>12</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>3</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>–4</td>
<td>16</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>–15</td>
<td>15</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>–30</td>
<td>12</td>
</tr>
</tbody>
</table>

The problem is to find allocations to the three users such that the total return is maximized. The following steps show the computations of backward recursion.

Stage 1

$S_1$: Amount of water available for allocation to User 3

$x_3$: Amount of water allocated to User 3

$x_3^*$: Allocation to User 3, that results in $f^*_1(S_1)$

$f^*_1(S_1)$: Maximum return due to allocation of $S_1$

$f^*_1(S_1) = \max \{ R_j(x) \}$

\[0 \leq x_3 \leq S_1 \]

\[0 \leq S_1 \leq 6\]

<table>
<thead>
<tr>
<th>S1</th>
<th>x3</th>
<th>R_j(x3)</th>
<th>$f^<em>_1(S_1) = x_3^</em>$ max { R_j(x3) }</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>0</td>
</tr>
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<td>12</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td></td>
<td>2</td>
</tr>
</tbody>
</table>

(Contd)
Stage 2

$S_2$: Amount of water available for allocation to Users 2 and 3 together.

$x_2$: Amount of water allocated to User 2

$S_2 - x_2$: Amount of water available for allocation at Stage 1 (to User 3)

$x_2^*$: Allocation to User 2 that results in

\[
\text{f}_{2,1}^*(S_2) = \max \left[ R_2(x_2) + f_1^*(S_2 - x_2) \right]
\]

\[0 \leq x_2 \leq S_2\]

\[0 \leq S_2 \leq 6\]

\[
\begin{array}{c|cc|c|c|c}
S_2 & x_2 & R_2(x_2) & f_1^*(S_2) & R_2(x_2) & f_2^*(S_2) \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 7 & 7 & 7 \\
1 & 1 & 5 & 0 & 0 & 5 \\
0 & 0 & 2 & 12 & 12 & 12 & 1, 2 \\
1 & 2 & 6 & 1 & 7 & 12 \\
2 & 6 & 0 & 0 & 6 & 6 \\
0 & 0 & 3 & 15 & 15 & 15 \\
3 & 1 & 5 & 2 & 12 & 17 \\
2 & 6 & 1 & 7 & 13 & 17 & 1 \\
3 & 3 & 0 & 0 & 3 & 3 \\
\end{array}
\]
(Contd)

<table>
<thead>
<tr>
<th>$S_2$</th>
<th>$x_2$</th>
<th>$R(x_1)$</th>
<th>$S_2 - x_2$</th>
<th>$f_1^1(S_2 - x_2)$</th>
<th>$R(x_1)$</th>
<th>$f_1^2(S_1) = \max {R(x_1) + f_1^1(S_2 - x_2)}$</th>
<th>$x_2$</th>
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</thead>
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<td>15</td>
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<td>2</td>
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<td>4</td>
<td>1</td>
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<td>3</td>
<td>0</td>
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</tr>
<tr>
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<td>15</td>
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<td>$-15$</td>
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<td>7</td>
<td>$-8$</td>
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<tr>
<td>6</td>
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<td>0</td>
<td>$-30$</td>
<td>0</td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

Stage 3

$S_3$: Amount of water available for allocation to Users 1, 2 and 3 together = 6 units

$x_1$: Amount of water allocated to User 1

$S_3 - x_1$: Amount of water available for allocation at Stage 2 (Users 2 and 3 together)

$x_1^*$: Allocation to User 1 that results in $f_1^*(S_3)$

$f_1^*(S_3) = \max \{R(x_1) + f_2^*(S_3 - x_1)\}$

<table>
<thead>
<tr>
<th>$S_3$</th>
<th>$x_1$</th>
<th>$R(x_1)$</th>
<th>$S_3 - x_1$</th>
<th>$f_2^1(S_3 - x_1)$</th>
<th>$R(x_1)$</th>
<th>$f_2^2(S_1) = \max {R(x_1) + f_2^1(S_3 - x_1)}$</th>
<th>$x_1^*$</th>
</tr>
</thead>
<tbody>
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</tbody>
</table>
We define, for forward recursion:

- $S_1$: Amount of water available for allocation at stage 1 (User 1);
- $S_2$: Amount of water available for allocation at stage 2 (Users 1 and 2 together);
- $S_3$: Amount of water available for allocation at stage 3 (Users 1, 2, and 3 together);

$f^*_1(S_1)$: maximum return due to allocation of $S_1$;

When the third (last) stage is solved, all the three users are considered for allocation. Thus the total maximum return is, $f^*_3(6) = 28$. The allocations to individual users are traced back as follows: From the table for Stage 3, we get $x_3 = 2$. From this, the water available at Stage 2 is obtained as $S_2 = Q - x_3 = 6 - 2 = 4$. We go to the table for Stage 2, with this value of $S_2 = 4$, and obtain the optimal value of $x_2$ as $x_2^* = 1$. From this we get the amount of water available for allocation at Stage 1 as $S_1 = S_2 - x_2^* = 4 - 1 = 3$. With this value of $S_1$, we enter the table for Stage 1, and obtain $x_1^* = 3$. Thus the optimal allocations are:

- $x_1^*$ = Allocation to User 1 = 2 units
- $x_2^*$ = Allocation to User 2 = 1 unit
- $x_3^*$ = Allocation to User 3 = 3 units

Maximum returns resulting from the allocations = 28.

**Forward Recursion** We will illustrate the forward recursion with the same example of water allocation.

In forward recursion the solution starts from the first user, User 1, and proceeds in the forward direction to the third user, User 3. At every stage in the forward recursion, we look backward to obtain optimal solutions from the other stages that have already been solved.

We define, for forward recursion:

- $S_1$: Amount of water available for allocation at stage 1 (to User 1);
- $S_2$: Amount of water available for allocation at stage 2 (to Users 1 and 2 together);
- $S_3$: Amount of water available for allocation at Stage 3 (to Users 1, 2, and 3 together);

$f^*_1(S_1)$: maximum return due to allocation of $S_1$;

Stage 1

\[
f^*_1(S_1) = \max \{ R_1(x_1) \mid 0 \leq x_1 \leq S_1, 0 \leq S_1 \leq 6 \}
\]

$S_1$: Amount of water available for allocation at stage 1 (User 1)

$x_1$: Amount of water allocated to User 1

$x_1^*$: Allocation to User 1 that results in $f^*_1(S_1)$
Systems Techniques in Water Resources

Stage 2

\[ f_2'(S_2) = \max \{R_1(x_2) + f_1'(S_2 - x_2)\} \]

\[ 0 \leq S_2 \leq 6 \]
\[ 0 \leq x_2 \leq S_2 \]

\( S_2 \): Amount of water available for allocation to Users 1 and 2 together.

\( x_2 \): Amount of water allocated to User 2.

\( x_2^* \): Allocation to User 2 that results in \( f_2'(S_2) \).

<table>
<thead>
<tr>
<th>( S_2 )</th>
<th>( x_2 )</th>
<th>( R_1(x_2) )</th>
<th>( f_1'(S_2 - x_2) )</th>
<th>( f_2'(S_2) = \max {R_1(x_2) + f_1'(S_2 - x_2)} )</th>
<th>( x_2^* )</th>
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(Contd)

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<th>( x_2 )</th>
<th>( R_1(x_2) )</th>
<th>( S_2 - x_2 )</th>
<th>( f_1'(S_2 - x_2) )</th>
<th>( f_2'(S_2) = \max {R_1(x_2) + f_1'(S_2 - x_2)} )</th>
<th>( x_2^* )</th>
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</table>
### Basics of Systems Techniques

#### (Contd)

\[S_2 \ x_2 \ R_2(S_2) \ x_2 - S_2 \ f_1(S_2 - x_2) + R_2(S_2) = \max \{R_2(S_2) + f_1(S_2 - x_2)\}\]

<table>
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<tr>
<th>(S_2)</th>
<th>(x_2)</th>
<th>(R_2(S_2))</th>
<th>(x_2 - S_2)</th>
<th>(f_1(S_2 - x_2))</th>
<th>(R_2(S_2))</th>
<th>(f_1(S_2 - x_2))</th>
<th>(S_2)</th>
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<td>8</td>
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</tr>
</tbody>
</table>

#### Stage 3

- **S₃**: Amount of water available for allocation to Users 1, 2 and 3 together = 6 units
- **x₃**: Amount of water allocated to User 3.
- **S₄**: Allocation to User 3 that results in \(f'_3(S_3)\).

\[f'_3(S_3) = \max \{R_3(S_3) + f'_3(S_3 - x_3)\}\]

\[S_3 = 6\]

\[0 \leq x_3 \leq S_3\]

<table>
<thead>
<tr>
<th>(S_3)</th>
<th>(x_3)</th>
<th>(R_3(S_3))</th>
<th>(x_3 - S_3)</th>
<th>(f_3(S_3 - x_3))</th>
<th>(R_3(S_3) + f'_3(S_3 - x_3))</th>
<th>(f'_3(S_3))</th>
<th>(S_3)</th>
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<td>0</td>
<td>0</td>
<td>12</td>
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</tbody>
</table>
The Bellman’s principle of optimality may be interpreted for backward recursion as: “No matter what state of what stage one may be in, in order for a policy to be optimum, one must proceed from that state and stage in an optimal manner,” and for forward recursion as: “No matter what state of what stage one may be in, in order for a policy to be optimum, one had to get to that state and stage in an optimal manner” (Loucks et al., 1981).

2.3.3 Characteristics of a DP problem

- A single \( n \)-variable problem is divided into \( n \) number of single variable problems. This requires that the objective function of the optimization problem be separable with respect to stages. For example, the function, \( R_1(X_1) + R_2(X_2) + \ldots + R_n(X_n) \) is separable, because we can identify exactly one variable in the objective function associated with each stage. Similarly the function, \( X_1 X_2 X_3 \ldots X_n \) is also separable. However, the function, \( X_1 X_2 + X_2 X_3 + X_3 X_4 \ldots \) is not separable and problems with such objective functions cannot be formulated as DP problems.
- A problem is divided into stages, with a policy decision required at each stage. In the water allocation problem, each user constitutes a stage, and the amount of water allocated at a stage constitutes a policy decision.
- Each stage has a number of possible states associated with it. In the water allocation problem, the amount of water available for allocation at a stage defines the state at that stage.
- The policy decision transforms the current state into a state associated with the next stage.
- A recursive relationship identifies the optimal decision at a given stage for a specific state, given the optimal decision for each state at the previous stage.
- A solution moves backward (or forward), stage by stage, till optimal decision for the last stage is found. From this solution, the optimal decisions for other stages are traced back.

Within this broad framework of discrete DP problems, we shall now discuss solutions for three specific problems in water resources decision making:
(a) Shortest Route Problem, (b) Reservoir Operation Problem, and (c) Capacity Expansion Problem.

2.3.4 Example Applications of DP

Shortest Route Problem

Consider the problem of determining the shortest route for a pipeline from among various possible routes available from destination to source. Figure 2.12 shows the network of possible routes connecting several nodes, along with the distance between two nodes, shown along the arrows, on a pipeline route. Starting from the source node A, the pipeline must run through one of the nodes B₁, B₂, or B₃, one of the nodes, C₁, C₂, or C₃, and one of the nodes, D₁, D₂, or D₃, before reaching the destination node E. The problem is to determine the shortest route between node A and node E. We may pose this problem as a multistage, sequential decision problem and solve the problem with Dynamic Programming. To do this, we define stages as shown in the figure. Starting from the source node A, we can reach one of the nodes B₁, B₂, or B₃ in Stage 1 and one of the nodes C₁, C₂, or C₃ in Stage 2, and one of the nodes D₁, D₂, or D₃ in Stage 3, and reach the destination node E in Stage 4. Using forward recursion, we define the state of the system, Sₙ, at a stage n (n = 1, 2, 3, 4) as the node reached in that stage, and the decision variable as the node xₙ₋₁ from which the node Sₙ is reached. We look for that node xₙ₋₁ out of all possible nodes xₙ, which results in the shortest distance from the source to the node Sₙ. We denote this node xₙ as xₙ* in Stage 1, the possible states, S₁, are B₁, B₂, and B₃. There is only one node from which any of these nodes could be reached, and that is the source node A itself, so that the decision variable, x₁, in this case is the source node A. Being the only node from which any of the nodes, B₁, B₂, and B₃, in Stage 1 may be reached, the node A also is x₁.
We proceed in a similar way to the other stages until Stage 4, and then trace back the solution to obtain the shortest route. The stagewise calculations are as follows:

Stage 1

- \( S_1 \): Node in Stage 1
- \( x_1 \): Node from which \( S_1 \) is reached
- \( d(x_1, S_1) \): Distance between nodes \( x_1 \) and \( S_1 \)
- \( \hat{f}(S_1) \): Minimum distance from source node \( A \) to node \( S_1 \)

\[
\hat{S}_1 = \min \{d(x_1, S_1)\}
\]

\[
\hat{x}_1 = \begin{cases} \hat{S}_1 & S_1 \in A \\ d(x_1, S_1) \end{cases}
\]

<table>
<thead>
<tr>
<th>( S_1 )</th>
<th>( x_1 = A )</th>
<th>( \hat{f}(S_1) )</th>
<th>( \hat{x}_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_1 )</td>
<td>10</td>
<td>10</td>
<td>( A )</td>
</tr>
<tr>
<td>( B_2 )</td>
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<td>15</td>
<td>( A )</td>
</tr>
<tr>
<td>( B_3 )</td>
<td>12</td>
<td>12</td>
<td>( A )</td>
</tr>
</tbody>
</table>

Stage 2

- \( S_2 \): Node in Stage 2
- \( x_2 \): Node in Stage 1 from which \( S_2 \) is reached
- \( d(x_2, S_2) \): Distance between nodes \( x_2 \) and \( S_2 \)
- \( \hat{f}(S_2) \): Minimum distance from source node \( A \) to node \( S_2 \)

\[
\hat{S}_2 = \min \{d(x_2, S_2) + \hat{f}(x_2)\}
\]

<table>
<thead>
<tr>
<th>( S_2 )</th>
<th>( x_2 )</th>
<th>( d(x_2, S_2) )</th>
<th>( \hat{f}(x_2) )</th>
<th>( d(x_2, S_2) + \hat{f}(x_2) )</th>
<th>( \hat{f}(S_2) )</th>
<th>( \hat{x}_2 )</th>
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<td>12</td>
<td>26</td>
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</tr>
</tbody>
</table>

Stage 3

- \( S_3 \): Node in Stage 3
- \( x_3 \): Node in Stage 2 from which \( S_3 \) is reached
- \( d(x_3, S_3) \): Distance between nodes \( x_3 \) and \( S_3 \)
- \( \hat{f}(S_3) \): Minimum distance from source node \( A \) to node \( S_3 \)

\[
\hat{S}_3 = \min \{d(x_3, S_3) + \hat{f}(x_3)\}
\]

<table>
<thead>
<tr>
<th>( S_3 )</th>
<th>( x_3 )</th>
<th>( d(x_3, S_3) )</th>
<th>( \hat{f}(x_3) )</th>
<th>( d(x_3, S_3) + \hat{f}(x_3) )</th>
<th>( \hat{f}(S_3) )</th>
<th>( \hat{x}_3 )</th>
</tr>
</thead>
<tbody>
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<td>26</td>
<td>( C_1 )</td>
<td></td>
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<td>9</td>
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<td>29</td>
<td>26</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( C_3 )</td>
<td>7</td>
<td>26</td>
<td>33</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(Contd)
In this table, values for \( f'(S_3) \) are read from the previous stage (Stage 2) table, for a given value of \( S_2 = x_3 \).

### Stage 4

- **State**: \( S_4 \)
- **Node in Stage 4**: \( E_1 \)
- **Node in Stage 3 from which \( S_4 \) is reached**: \( x_4 \)
- **Distance between \( x_4 \) and \( S_4 \)**: \( d(x_4, S_4) \)
- **Minimum distance from source node \( A \) to node \( S_4 \)**: \( f_4(S_4) = \min \{ d(x_4, S_4) + f_3(x_4) \} \)

<table>
<thead>
<tr>
<th>( S_4 )</th>
<th>( x_4 )</th>
<th>( d(x_4, S_4) )</th>
<th>( f_4(x_4) )</th>
<th>( d(x_4, S_4) + f'_3(x_4) )</th>
<th>( f'_4(S_4) )</th>
<th>( x'_4 )</th>
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<td>30</td>
<td>30</td>
<td>( C_3 )</td>
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<td>41</td>
<td>30</td>
<td>30</td>
<td>( C_2 )</td>
</tr>
</tbody>
</table>

The optimal route is traced back from the last table as follows: The table shows that to reach the destination node \( E \), we must come from the node, \( x_4 = D_1 \) in Stage 3. We enter the table for Stage 3 with \( S_3 \) (node to be reached in Stage 3) = \( C_1 \). Similarly from Stage 2 table, we get \( x_3 = C_1 \) and from Stage 1 table, \( x_2 = A \). The shortest route is thus, \( A-B_1-C_1-D_1-E \), and the shortest distance is 35 units, which is the optimized objective function value, \( f'_4(S_4) \), in the last stage.

### Reservoir Operation Problem

A classical multistage decision problem in the area of water resources management is the reservoir operation problem. Simply stated, the problem is to specify release \( R_t \) during a time period \( t \) (such as a month, a season, etc), when the storage, \( S_t \), at the beginning of the period \( t \), and the inflow, \( Q_t \), during the period \( t \) are known. The sequence of releases, \( \{ R_t \} \), in a year, is called the Reservoir Release Policy for the year. The release policy is obtained to optimize (minimize or maximize) a system performance measure (e.g., hydropower produced during the year, annual deviations of irrigation allocations from targets etc.), which is, in general, a function of the storage, \( S_t \), and the release \( R_t \). Many variations of this simple problem, with varying degrees of complexities, are discussed in Chapters 5 to 9.
The reservoir operation problem may be solved as a multistage decision problem using DP. The time period \( t \) for which the release decision \( R_t \) is to be made defines a stage in the DP. The storage, \( S_t \), at the beginning of the period \( t \) is the state variable and \( R_t \), the release during the period \( t \) is the decision variable. Further, only discrete values, (such as 0, 10, 20, \ldots) are considered for storage, inflow and release.

The storage, \( S_t \), at the beginning of the period \( t \) changes to storage \( S_{t+1} \) at the beginning of period \( t+1 \) (or storage at the end of period \( t \)) because of the inflow \( Q_t \) and the release \( R_t \) (see Figure 2.13). This transformation is governed by the reservoir mass balance, or the storage continuity equation, which is written, neglecting all losses, as

\[
S_{t+1} = S_t + Q_t - R_t
\]

The reservoir storage must also satisfy the capacity constraint, \( S_t \leq K \), where \( K \) is the live storage capacity of the reservoir.

![Reservoir Operation Problem](image)

The operational objective may be to maximize the total net benefit during a year, which is stated as

\[
\text{Maximize} \sum_{t=1}^{T} B_t(S_t, R_t)
\]

where \( B_t(S_t, R_t) \) is the net benefit during period \( t \) for given values of \( S_t \) and \( R_t \), and \( T \) is the number of periods in the year. The problem is illustrated here for derivation of reservoir operating policy when only the current year is of interest, with the initial storage, \( S_1 \), specified. Derivation of stationary policy using DP is discussed in Chapter 5.

Starting with the last period, \( T \), in the year, we solve this problem with backward recursion. Letting \( n \) denote the stage in the DP, we define \( f^*_n(S_t) \) as the maximized net benefits up to and including the period \( t \). Note that the period \( t \) corresponds to stage \( n \) in this notation. The index \( n \) keeps track of the
stage in the DP and the index \( t \) denotes the time period within the year. The corresponding release in period \( t \), which results in \( f^*_t(S_t) \), is denoted as \( R'_t \).

In Stage 1, \((n = 1 \text{ and } t = T)\), we solve the problem only for one period and obtain \( f^*_1(S_1) \) for possible values of \( S_1 \) as

\[
f^*_1(S_1) = \max \{ B_1(S_1, R_1) \} \quad \text{subject to} \quad 0 \leq R_1 \leq S_1 + Q_1,
\]

The two constraints, \( 0 \leq R_1 \leq S_1 + Q_1 \), and \( S_1 + Q_1 = R_1 \leq K \), specify the feasible values for the release \( R_1 \), over which the search is made. The first one restricts the release to the total water available in storage in period \( t \), and the second one ensures that the end of period storage is restricted to the live storage capacity.

In Stage 2, \( n = 2 \), \( t = T - 1 \)

\[
f^*_{T-1}(S_{T-1}) = \max \{ B_{T-1}(S_{T-1}, R_{T-1}) + f^*_{T}(S_{T-1}) \} \quad \text{subject to} \quad 0 \leq R_{T-1} \leq S_{T-1} + Q_{T-1},
\]

The term, \( S_{T-1} + Q_{T-1} = R_{T-1} \), appearing as the argument in \( f^*_{T}(\cdot) \), is the storage at the end of the period \( T - 1 \), which is also the storage at the beginning of the period \( T \) for which the solution has been already obtained in Stage 1. In the second stage, we seek to obtain the release \( R_{T-1} \), when the storage \( S_{T-1} \) is known, such that the total benefit up to the end of the year (i.e. for periods, \( T - 1 \) and \( T \) together) is maximized. For a specified value of \( S_{T-1} \), the first term, \( B_{T-1}(S_{T-1}, R_{T-1}) \), denotes the benefits for the period \( T - 1 \) corresponding to the decision \( R_{T-1} \), and the second term, \( f^*_{T}(S_{T-1} + Q_{T-1} - R_{T-1}) \) gives the maximum benefit corresponding to the storage at the beginning of period \( T \), resulting from the decision \( R_{T-1} \).

Proceeding in a similar way, the general recursive equation for any period \( t \) is written as

\[
f^*_t(S_t) = \max \{ B_t(S_t, R_t) + f^*_{t+1}(S_{t+1} + Q_t - R_t) \} \quad \text{subject to} \quad 0 \leq R_t \leq S_t + Q_t, \quad S_t + Q_t - R_t \leq K.
\]

At the last stage, \( t = 1 \) (\( n = T \)), the storage, \( S_1 \), is specified to be \( S' \). The recursive equation is solved only for this specified value of \( S_1 \), and is written as

\[
f^*_1(S_1 = S') = \max \{ B_1(S', R_1) + f^*_{T}(S' + Q_1 - R_1) \} \quad \text{subject to} \quad 0 \leq R_1 \leq S' + Q_1, \quad S' + Q_1 - R_1 \leq K.
\]

The solution is traced back after the last stage calculations as shown in Fig. 2.14. Starting with the known storage, \( S' \), at the beginning of period \( t = 1 \), the optimal release \( R'_1 \), obtained in the last stage calculations defines the storage at the beginning of period \( t = 2 \), as, \( S_2 = S' + Q_1 - R'_1 \). With this value of \( S_2 \), the optimal release \( R'_2 \) is obtained from the previous stage (period \( t = 2 \)) calculations. This process is continued until optimal releases and corresponding storages for all periods are obtained.
The following numerical example illustrates the solution procedure for the reservoir operation problem.

**Example 2.3.3** Inflows during four seasons to a reservoir with storage capacity of 4 units are, respectively, 2, 1, 3, and 2 units. Only discrete values, 0, 1, 2, … , are considered for storage and release. Overflows from the reservoir are also included in the release. Reservoir storage at the beginning of the year is 0 units. Release from the reservoir during a season results in the following benefits which are same for all the four seasons.

<table>
<thead>
<tr>
<th>Release</th>
<th>Benefits</th>
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<tr>
<td>0</td>
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<tr>
<td>1</td>
<td>250</td>
</tr>
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<td>2</td>
<td>320</td>
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<tr>
<td>6</td>
<td>410</td>
</tr>
<tr>
<td>7</td>
<td>120</td>
</tr>
</tbody>
</table>

To obtain the release policy we use backward recursive equation, starting with the last stage.

\[ s_1 = s_4 + Q_1 - R_1 \]

\[ s_2 = s_2 + Q_2 - R_2 \]

\[ s_3 = s_3 + Q_3 - R_3 \]

\[ s_4 = s_4 + Q_4 - R_4 \]

With \( t = 4 \) at \( n = 1 \), we seek a solution for all possible values of \( S_4 \), the storage at the beginning of period (season) 4. These possible values are 0, 1, 2, 3, and 4, the storage capacity being 4 units. Inflow during the period being 2 units, some values of release will be infeasible if the end-of-period storage resulting from such values is more than 4 units. For example, for \( S_4 = 3 \), release \( R = 0 \) is infeasible, because, \( S_4 + Q_4 - R = 3 + 2 - 0 = 5 \) is more than the capacity, 4. The following tables give the calculations for each stage.

**Stage 1:** \( t = 4; \ n = 1 \)

\[ Q_4 = 2 f_4^1 (S_4) = \max \left[ R_l | R_l \right] \]

\[ 0 \leq R_1 \leq S_4 + Q_4 \]

\[ S_4 + Q_4 - R_4 \leq 4 \]
Stage 2: \( t = 3; n = 2 \)

\( Q_1 = 3 \sum \frac{f}{S_3} (S_3) = \max \{ B_3(R_3) + \sum f_3(S_3 + Q_3 - R_3) \} \)

\( 0 \leq R_3 \leq S_3 + Q_3 \)

\( S_3 + Q_3 - R_3 \leq 4 \)

<table>
<thead>
<tr>
<th>( S_3 )</th>
<th>( R_3 )</th>
<th>( R_3(R_3) )</th>
<th>( f_3(S_3) )</th>
<th>( R_3' )</th>
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</tbody>
</table>

(Contd)
Systems Techniques in Water Resources

(Cord)

Stage 3: \( t = 2; n = 3 \)

\[ Q_2 = 1 \quad f_1^1(S_2) = \max \left[ B_2(R_2) + f_1^1(S_2 + Q_2 - R_2) \right] \]

\[ 0 \leq R_2 \leq S_2 + Q_2 - R_1 \leq 4 \]

<table>
<thead>
<tr>
<th>( S_2 )</th>
<th>( R_2 )</th>
<th>( B_2(R_2) )</th>
<th>( S_2 + Q_2 - R_2 )</th>
<th>( f_1^1(S_2 + Q_2 - R_2) )</th>
<th>( f_1^1(S_2) )</th>
<th>( R_2^* )</th>
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<td>800</td>
<td>1520</td>
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</tbody>
</table>

Stage 4: \( t = 1; n = 4 \)

\[ Q_1 = 2 \quad f_1^1(S_1) = \max \left[ B_1(R_1) + f_1^1(S_1 + Q_1 - R_1) \right] \]

\[ 0 \leq R_1 \leq S_1 + Q_1 - R_1 \leq 4 \]

<table>
<thead>
<tr>
<th>( S_1 )</th>
<th>( R_1 )</th>
<th>( B_1(R_1) )</th>
<th>( S_1 + Q_1 - R_1 )</th>
<th>( f_1^1(S_1 + Q_1 - R_1) )</th>
<th>( f_1^1(S_1) )</th>
<th>( R_1^* )</th>
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<td>0</td>
<td>800</td>
<td>1520</td>
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</tbody>
</table>
Trace back:
\( R_1^1 = 1 \), from the last table.
\( S_2^1 = S_1 + Q_1 - R_1^1 = 0 + 2 - 1 = 1 \)
\( R_2^1 = 1 \) from Stage 3 table, corresponding to \( S_2^1 = 1 \)
\( S_3^1 = S_2^1 + Q_2 - R_2^1 = 1 + 1 - 1 = 1 \)
\( R_3^1 = 3 \) from Stage 2 table.
\( S_4^1 = S_3^1 + Q_3 - R_3^1 = 1 + 3 - 3 = 1 \)
\( R_4^1 = 2 \) from Stage 1 table.

The optimal release sequence for this problem, thus, is \((1,1,3,2)\) during the four periods. The maximum net benefits that result from this release policy is the optimized objective function value, \( f^1(S_1) = 1460 \) units.

Note: In some situations we may have multiple solutions. In this example, \( S_2 = 2 \) and \( S_4 = 4 \) both have two solutions each, and if either of them (\( S_2 = 2 \) or \( S_4 = 4 \)) were to appear in the trace back, both the solutions must be separately traced back. Such a situation yields alternate optimal policies, all resulting in the same (optimal) objective function value.

**Capacity Expansion Problem**

Water resource decision makers often face problems related to investments on expansion of existing capacities of either an entire system or individual components of a system. The decisions deal with investments at different times, starting with the present time, on capacity expansion needed over the years in future. A typical problem in this class of problems is to decide in what steps the expansion over the next \( n \) years should be carried out so that the present worth of the investment is a minimum. For example, we may need to decide on the investment on capacity expansion during next 5 years, 5 years after that and so on, if the planned expansion of the capacity is to be achieved after, say, 25 years (see Fig. 2.15). Since we decide on investments that actually need to be made only in future, a great deal of uncertainty (mainly with respect to economics, but also due to the actual capacity needed at different times in future) is associated with such investment decisions. Actual implementation of the decisions therefore is to be done in an adaptive manner by updating the decisions as and when new information becomes available. For the purpose of the discussion here, we ignore the uncertainty aspects.

To solve the capacity expansion problem with dynamic programming, we define the following terms:

- \( S_t \): Existing capacity at the beginning of period \( t \).
- \( x_t \): Expansion in period \( t \).
- \( C_t(S_t, x_t) \): Discounted present worth of cost of expansion in period \( t \).
- \( D_t \): Required minimum capacity at the end of period \( t \).
$D_T$ : Capacity required at the end of planning horizon (time span over which the capacity expansion is planned starting from the existing capacity in the present time).

$f_t(S_t)$ : Minimum discounted present worth of expansion from beginning of period $t$ up to the end of the last period $T$.

The time period $t$ may be typically 5–10 years depending on the size and nature of the expansion planned. The discounted present worth, $C_t(S_t, x_t)$, denotes the present worth of the cost incurred in the future time period $t$, and is, in general, a function of the capacity $S_t$ at the beginning of period $t$, the expansion $x_t$ during the period $t$, and also the discount rate for obtaining the present worth of the cost incurred at the future time $t$. Discounting techniques to obtain worth of money at different times are discussed in the Chapter 3. Here, it is assumed that discounted present worth of investment made in future time $t$ is available.

The time period $t$ constitutes a stage in the DP. The state variable is $S_t$, the capacity existing at the beginning of the period $t$, and the decision variable is $x_t$, the capacity expansion during the period $t$. Starting with the capacity $S_1$ at the beginning of period 1, an expansion of $x_t$ during the period $t$ results in capacity $S_{t+1}$ at the beginning of period $t+1$. The state transformation equation is thus, $S_{t+1} = S_t + x_t$. The capacity at the end of the period $t$ must be at least equal to the required capacity, $D_t$, i.e. $S_{t+1} \geq D_t$. The state variable $S_t$ can take values from $D_t - 1$ (the required capacity at the end of the period $t - 1$ or at the beginning of period $t$) to $D_t$, the required capacity at the end of the planning horizon. The range of values for $x_t$ will be 0 to $D_t - S_t$. We seek decisions on the steps in which the planned capacity $D_T$ must be achieved starting with the known existing capacity at the beginning of the first period, $S_1$. We solve this problem using backward recursion, as shown in Fig. 2.16.

At the beginning of period $T$, the minimum available capacity is $D_{T-1}$. Therefore, the possible values that the state variable $S_T$ may assume are $D_{T-1} \leq S_T \leq D_T$. Since this is the last time period in the planning horizon, we write
Basics of Systems Techniques

\[ f_T(S_T) = \min \left[ C_T(S_T, x_T) \right] \]

\[ D_{T-1} \leq S_T \leq D_T \]

\[ s_T = D_T - S_T \]

Note that since the capacity \( D_T \) must be achieved at the end of the period \( T \), \( x_T \) is exactly equal to \( (D_T - S_T) \).

In the next stage, period \( T - 1 \),

\[ f_{T-1}(S_{T-1}) = \min \left[ C_{T-1}(S_{T-1}, x_{T-1}) + f_T(S_{T-1} + x_T) \right] \]

\[ D_{T-2} \leq S_{T-1} \leq D_T \]

\[ 0 \leq x_{T-1} \leq D_T - S_{T-1} \]

In general, for any period \( t \),

\[ f_t(S_t) = \min \left[ C_t(S_t, x_t) + f_{t+1}(S_t + x_t) \right] \]

\[ D_{t-1} \leq S_t \leq D_t \]

\[ 0 \leq x_t \leq D_t - S_t \]

In the last stage, \( t = 1 \), the capacity \( S_1 \) is known, and the general recursive equation is solved only for that known value of \( S_1 \). The optimal solution is then traced back using the state transformation equation.

The following numerical example illustrates the solution procedure.

**Example 2.3.4** A city water supply project envisages expansion of the storage system from the existing capacity of 100 units to 200 units in the next 20 years. The additional capacity required at the end of each of the 5 years and the discounted present worth for additional capacities are as given below:

<table>
<thead>
<tr>
<th>Time</th>
<th>Required Additional Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>End of 5th year</td>
<td>20</td>
</tr>
<tr>
<td>End of 10th Year</td>
<td>40</td>
</tr>
<tr>
<td>End of 15th year</td>
<td>60</td>
</tr>
<tr>
<td>End of 20th year</td>
<td>100</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Discounted present worth of cost</th>
<th>Additional Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0 20 40 60 80 100</td>
</tr>
<tr>
<td>( t ) Period (Years)</td>
<td></td>
</tr>
<tr>
<td>1 1-5</td>
<td>0 120 150 200 250 280</td>
</tr>
<tr>
<td>2 6-10</td>
<td>0 80 110 130 150</td>
</tr>
<tr>
<td>3 11-15</td>
<td>0 60 80 100</td>
</tr>
<tr>
<td>4 15-20</td>
<td>0 40 50</td>
</tr>
</tbody>
</table>

Note that since the minimum additional capacity created at the end of the first 5 years is 20, only a maximum of 80 additional units need to be created during the remaining periods. The cost of additional capacity is therefore shown only for a maximum of 80 units for years 6–10. Similarly, for the other
years, costs are shown only up to the maximum capacity that still needs to be created till the end of the planning horizon. Starting with the last period, T (15–20 years), we solve the problem using backward recursion. The computations are shown in the following tables.

Stage 1: \( t = 4 \)
\[
f_4(S_4) = \min \left\{ C_4(x_4) \right\}
\]
\[160 \leq S_4 \leq 200\]
\[S_4 + x_4 = 200\]

<table>
<thead>
<tr>
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<th>( x_4 )</th>
<th>( C_4(x_4) )</th>
<th>( f_4(S_4) )</th>
<th>( x_4 )</th>
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<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Stage 2: \( t = 3 \)
\[
f_3(S_3) = \min \left\{ C_3(x_3) + f_4(S_3 + x_3) \right\}
\]
\[140 \leq S_3 \leq 200\]
\[160 \leq S_3 + x_3 \leq 200\]

<table>
<thead>
<tr>
<th>( S_3 )</th>
<th>( x_3 )</th>
<th>( C_3(x_3) )</th>
<th>( S_3 + x_3 )</th>
<th>( f_3(S_3 + x_3) )</th>
<th>( f_3(S_3) )</th>
<th>( S_3 )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>40</td>
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<td>160</td>
<td>50</td>
<td>50</td>
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<tr>
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<td>200</td>
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<td>0</td>
</tr>
</tbody>
</table>

Stage 3: \( t = 2 \)
\[
f_2(S_2) = \min \left\{ C_2(S_2, x_2) + f_3(S_2 + x_2) \right\}
\]
\[120 \leq S_2 \leq 200\]
\[140 \leq S_2 + x_2 \leq 200\]

<table>
<thead>
<tr>
<th>( S_2 )</th>
<th>( x_2 )</th>
<th>( C_2(x_2) )</th>
<th>( S_2 + x_2 )</th>
<th>( f_3(S_2 + x_2) )</th>
<th>( f_2(S_2) )</th>
<th>( S_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>120</td>
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</table>

(Contd)
The problems we dealt with so far in this chapter had only one state variable. In most practical problems, we may have to treat more than one variable to describe the state of the system at a stage. Consider the problem of irrigation water allocation among a number of crops where, apart from water available, land available for irrigation is also a state variable. The allocation problem thus deals with allocation of two resources among the crops: land and water. The total water available for irrigation is limited to $Q$, and the total available land is limited to an area $A$. Assuming that each unit of land requires an amount $w_j$ for crop $j$ ($j = 1, 2, \ldots, n$), and that the returns from allocating $x_j$ units of water to crop $j$ is $R_j(x_j)$, we may write the optimization problem as

$$\max \sum_{j=1}^{n} R_j(x_j)$$

subject to
\[ \sum_{j=1}^{n} x_j \leq Q \text{ (water availability constraint),} \]
\[ \sum_{j=1}^{n} x_j/w_j \leq A \text{ (land availability constraint)} \]

Note that \( x_j/w_j \) is the land allocated to crop \( j \). With non-negativity of variables included, this problem may be solved as a linear programming problem.

To formulate this problem as a DP problem, we need to define two state variables: \( S_j \), the water available at stage \( j \), and \( L_j \), the land available at stage \( j \).

The backward recursion may be written as
\[
f_j(S_j, L_j) = \max \{ R_j(x_j) + f_{j+1}(S_j - x_j, L_j - x_j/w_j) \}
\]
\[ 0 \leq x_j \leq S_j, \quad x_j/w_j \leq L_j \]

This recursive equation should be solved for all values of \( S_j \) and \( L_j \) satisfying \( 0 \leq S_j \leq Q \) and \( 0 \leq L_j \leq A \).

While solving DP problems on computers, with two state variables, therefore, we need to store two-dimensional vectors for all the stages, as they are required for tracing back the solution after computations for all stages are completed. In general, for a \( k \)-state variable problem, we need to store a \( k \) dimensional vector in the computer memory, and as the number of state variables increases, the computer memory requirements increase rapidly. This is called the *Curse of Dimensionality* of DP. Till recently, most common DP applications dealt with only a few (3 or 4) state variables. With rapid progress in the computer speed and memory availability in recent years, however, problems with more than 6 state variables have also been solved using DP (e.g. Mujumdar and Ramesh, 1997).

**Problems**

2.3.1 Solve the following 4-user water allocation problem to maximize the total returns, using both forward and backward recursion of dynamic programming:

Water available for allocation = 60 units, to be allocated in discrete units of 0, 10, 20 … 60. Returns from the four users for a given allocation, are given in the table below:

<table>
<thead>
<tr>
<th>Allocation</th>
<th>User 1</th>
<th>User 2</th>
<th>User 3</th>
<th>User 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-3</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>20</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>30</td>
<td>6</td>
<td>4</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>40</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>50</td>
<td>3</td>
<td>6</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>60</td>
<td>3</td>
<td>7</td>
<td>0</td>
<td>10</td>
</tr>
</tbody>
</table>

(Ans: User 1: 10; User 2: 10; User 3: 10; User 4: 30 Max. return: 17 units)
2.3.2 A pipeline is to be laid between node G and node C shown in the figure below. The pipeline can pass only along the routes shown by solid lines between intermediate nodes in the figure. The distance between two nodes is shown on the line joining the two nodes. Obtain the shortest distance for the pipeline using dynamic programming. (Hint: Define stages along diagonal lines, B–F, A–E–J etc., and use backward recursion from C to G).

(Ans: G–D–A–B–C ; Shortest distance: 42 units)

2.3.3 Solve the reservoir operation problem discussed in Sec. 2.3.5 with the initial storage, \( S_1 \), equal to the reservoir capacity of 4 units. Compare the release policy with that obtained in the example discussed in the text.

2.3.4 Solve the same reservoir operation problem (as in Sec. 2.3.5) with the storage at the end of the year (that is, at the end of the last season, \( t = 4 \)) specified as 0 units, and with no constraint on the initial storage. (Hint: Note that in backward recursion, while solving for \( t = 4 \), you must consider only those releases which result in the end of season storage of 0 units).

2.3.5 Solve the reservoir operation problem by combining the constraints in Problems 2.3.3 and 2.3.4 (that is, initial storage = 4 units, and end of year storage = 0 units). Compare the net benefits resulting from the three solutions with that of the problem discussed in the text.

2.3.6 A town decides to expand its water supply system with an existing capacity of 10 units to the ultimate requirement of 40 units by the end of 15 years from now, in stages of 5 years each. The present worth of cost of capacity expansion at any stage is estimated to be equal to the square of the number of units added at that stage. The capacity requirement is estimated to be 15, 25 and 40 units by the end of 5, 10 and 15 years from now. Determine how many units should be added at each stage for minimum total cost of capacity expansion over a
15 year planning horizon. Capacity can be added only in 5 unit increments.

\[(\text{Ans: } 5, 10, 15; \text{ Present worth: } 173.95)\]

2.3.7 Determine how many projects of type A, B and C should be built in order to maximize the total net return from all projects, given the following data.

<table>
<thead>
<tr>
<th>Project Type</th>
<th>Cost of each project</th>
<th>Net return/project</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2000</td>
<td>280</td>
</tr>
<tr>
<td>B</td>
<td>3000</td>
<td>440</td>
</tr>
<tr>
<td>C</td>
<td>4000</td>
<td>650</td>
</tr>
</tbody>
</table>

Total amount available for all projects = 24000 units. At least one project of each type should be built. Not more than 4 projects of each type should be built.

\[(\text{Ans: A—1 project; B—2 projects; C—4 projects. Max. net return: 3760 units})\]

2.4 SIMULATION

Simulation is a modelling technique, which is often used to examine and evaluate the performance of complex water resources systems. It is particularly useful where optimization techniques cannot be used because of their limitations. Simulation, by itself, is not an optimization technique, but can be articulated to yield near optimum results. In water resources modelling, knowledge of solutions around the optimum are just as useful as the optimum itself, from a practical point of view.

Simulation is by far the most widely used method for evaluating alternate water resources systems and plans. Though it is not an optimizing procedure, for a given set of design and policy parameters, it offers a rapid means of evaluating the expected performance of the system on the computer using computer programs written exclusively for the specific problem under consideration. For example, one can simulate the performance of a reservoir for 50 years, based on given operating rules, to determine the sequence of annual irrigation and hydropower benefits. The operating rules may have been obtained as a solution of an optimization model for the system in consideration (e.g. steady state reservoir operating policy derived from a stochastic DP model for the system). Also, for given aquifer parameters, simulation can be used to determine the changes in the groundwater levels over a period of time for different pumping patterns. The governing equations of groundwater flow have to be known and suitable computer programs developed for use in simulation. In other words, the operating policy and the system parameters must be known for making a simulation run, while either of them can be iteratively determined through multiple simulation runs. Simulation is an ideal tool for performance evaluation.
2.4.1 Components of a Simulation Model

Inputs, physical relationships and constraints, operating rules, and outputs constitute the components of a simulation model. Inputs are those that essentially “drive” the model. The model transforms the inputs into outputs according to a set of physical or governing relationships. In reservoir simulation, for example, reservoir inflow, evaporation rate, and irrigation water demand are among the inputs required for simulation.

Physical relationships and constraints define the relationships among the physical variables of the system: e.g. Reservoir storage–elevation–area relationships, storage continuity relationships, and soil moisture balance.

Operating rules define how the system is operated: e.g. Reservoir release policies, rule curves.

Outputs are a measure of system response resulting from operating the system following known or specified rules and constraints: e.g. Quantum of reservoir release for irrigation, hydropower, low flow augmentation, etc.

2.4.2 Steps in Simulation

Once it is decided to use simulation to study a system, the first step is to decompose the system into components or subsystems, which are held together by linkages. Each subsystem and its linkages are tagged on with specific operating rules and constraints. Computer programs are formulated for each of the subsystems and for flow of information through the linkages. The next important step is model verification. This is carried out with known inputs and outputs for each subsystem, to verify that simulation of the total system produces the known outputs from the given set of inputs. The model is now ready to take on additional or alternate sets of inputs and give the corresponding outputs resulting from simulation.

Simulation is extensively used in the analysis of complex water resources systems, both for solving direct and inverse problems. In reservoir operation, for example, simulation can be used to determine annual hydropower generation, flood peak attenuation, irrigation allocation (direct problems), or to determine the operating policy, or rule curves (inverse problems). Determination of aquifer performance through numerical solution of groundwater flow equations (direct problem), determination of aquifer parameters from known inputs and outputs (inverse problem) are typical examples for which simulation is extensively used in groundwater hydrology.

2.4.3 Simulation Runs

Each simulation run for a given set of inputs (design and operating policy) results in a trace of the system performance output. The results from a number of such runs define what is referred to as the response surface. If the response surface is steep, it indicates that the system is highly sensitive to variations in input, and vice versa. If the system is highly sensitive, a relatively large number of simulation runs is necessary to capture the near optimum region. Global optimum can never be guaranteed in simulation.
Generally, for many water resources problems in practice, the response surfaces will be relatively smooth and flat near the optimum. Because of the possible underlying danger of identifying only a local optimum solution in simulation, the experience and judgement of the systems analyst play a major role in interpreting the usefulness and significance of the solution obtained from these techniques.

2.5 COMBINATION OF SIMULATION AND OPTIMIZATION

Most water resources systems are large in size and quite complex to be modelled entirely as optimization models. Often, the use of a single algorithm is therefore either not adequate or computationally intractable. This necessitates the use of a combination of two or more models. The combination of simulation–optimization (S–O) is very commonly used in modelling. A major advantage of the S–O methodology in most situations is that the physical processes such as the mass, energy, and temperature balance are accounted through simulation outside the optimization model, thus reducing the size and complexity of the optimization model itself. Such modelling situations arise especially in management of water quality where the transport of pollutants across a stream is modeled by a simulation model reproducing the physical processes, and the result from such a simulation model is used in the optimization model to evaluate the objective function value (see Fig. 2.17).

In some situations it is advantageous to first carry out a detailed simulation of the system to identify initial approximate solutions and then refine these solutions through optimization. Such cases often arise in large multireservoir systems planning and operation problems using nonlinear optimization models (e.g. Vijay Kumar et al., 1996). Most algorithms for solving nonlinear optimization problems need an initial solution, and the speed of convergence to an optimal solution depends on the particular initial solutions provided. By providing near-optimal initial solutions (identified through simulation) the computational time for optimization may be significantly reduced.

A common approach used for performance evaluation of water resources systems also involves a combination of optimization and simulation models.
For example, in problems dealing with reservoir systems, an optimal release policy is first determined with an optimization model, and the system behavior under the optimal policy is simulated over a long period of time to examine the performance through indicators (e.g. Hashimoto et al., 1982) such as reliability of meeting demands, and resiliency of the system to recover from failure when one occurs.

REFERENCES


Further Reading

Economic Considerations in Water Resources Systems

3.1 BASICS OF ENGINEERING ECONOMICS

3.1.1 General Principles

In this section, we discuss the value of money at different points of time and, based on it, a methodology of comparing alternative plans on an economic basis.

The first and foremost thing to realize is that money has time value. The same amount of money is worth more today than tomorrow. Money can be put to productive use by way of investment. Comparison of amounts of money must be made based on a common time reference.

Interest may be thought of as the price one has to pay for money borrowed from someone else, or as the return derived from capital invested, or as the reward for making available money for someone who needs it. Because of the interest rate, a given amount of money today has a higher value when productively invested than the same amount in future.

If one unit of money were invested in a bank at a nominal interest rate, \( i \) (expressed as % per year by default), it would accumulate in one year to \((1 + i)\). However, if the money is compounded \( m \) times in a year (i.e. if there are \( m \) compounding periods in a year), the amount accumulated is calculated as follows:

Let there be \( m \) periods in a year and let the money be compounded at the end of each period. The interest rate per period is \( \frac{i}{m} \). If Re 1 is invested in a bank, it would accumulate in a year to \((1 + \frac{i}{m})^m\). The effective annual interest rate then is \((1 + \frac{i}{m})^m - 1\), or Effective rate of interest, \( i_e = (1 + \frac{i}{m})^m - 1\).
Example 3.1.1 The nominal rate of interest is 10%. Determine the effective rate of interest when money is compounded

1. yearly
2. half yearly
3. quarterly
4. daily

1. yearly: $i_e$ is the same as $i = 10\%$

2. No. of corresponding periods in a year = 2
   
   Interest rate per 6 months $= \frac{i}{2} = \frac{10}{2} = .05$
   
   $\therefore$ One rupee becomes $(1.05)^2$ at the end of the year $= 1.1025$

   $\therefore$ Effective rate of interest $= 1.1025 - 1 = .1025$ or 10.25%

3. No. of compounding periods = 4
   
   Interest rate per period $= \frac{i}{4} = \frac{10}{4} = .025$
   
   One rupee becomes $(1.025)^4 = 1.10381$
   
   $\therefore$ Effective interest rate $= 1.10381 - 1 = .10381$ or 10.381%

4. No. of compounding periods $= m = 365$
   
   Interest rate per day $= \frac{i}{m} = \frac{10}{365}$
   
   $\therefore$ Effective rate of interest $= \left(1 + \frac{10}{365}\right)^{365} - 1 = 1.10516 - 1$
   
   $\therefore$ Effective rate of interest $= .10516$ or 10.516%

Discount rate is the interest rate used in discounting future cash flows. It is an expression of the time value of capital used in equivalence calculations comparing alternatives, and is essentially a value judgment based on a compromise between present consumption and capital formation from the viewpoint of the decision maker (James and Lee, 1971).

Depreciation is the consumption of investment in property, or is the reduction in the value of property due to its decreased ability to perform present and future service.

Sinking fund is a fund created by making periodic (usually equal) deposits at compound interest in order to accumulate a given sum at a given future time for some specific purpose. If we buy a pump for Rs 10,000, which has a service life of say 10 years with no salvage value (salvage value of a property is the net worth of money that can be realized at the end of its service life), then to compensate for the depreciation of the pump with time, we raise a sinking fund by paying equal amounts of money at regular intervals so that the total sum (with compound interest) accumulates to Rs 10,000 at the end of 10 years at the specified discount rate.
If the salvage value (market value at the end of the service life) of the pump at the end of 10 years is \( L \), then the sinking fund to be raised is the cost of the pump, \( P \), minus the salvage value, \( L \), i.e., \((P - L)\) ignoring inflation. We can buy another new pump at the end of 10 years with this sum, \((P - L)\). We ignore the effect of inflation throughout our discussion on economic analysis. Sometimes sinking fund is simply computed based on the straight-line depreciation method. The annual depreciation, in this method, is calculated as the ratio of the total depreciable value, \((P - L)\) to the service life, \( n \) in years. But the most accurate method of computing the sinking fund is through the use of the sinking fund factor, defined later in this section under discount factors, which precisely accounts the time value of money.

### 3.1.2 Discount Factors

Discounting refers to translating future money to its present value, and compounding refers to translating present money to a future period in time.

For equivalence calculation of money at different points of time and for comparison of engineering alternatives, we need to know the commonly used discounting factors:

The following notation is used in this section.

- \( i \) = interest rate per year expressed in per cent
- \( P \) = present sum of money
- \( F \) = future sum of money
- \( A \) = annual payment, or payment per period

Usually, three values out of these four would be known and it will be required to compute the fourth.

A cash flow diagram is usually drawn in each case, depicting the payments on one side and receipts on the other side of a horizontal time line, in order to facilitate computations of the value of money at different times.

#### 1. Single Payment Factors

(a) **Compound Amount Factor** \((F/P, i\%, n)\) This factor multiplied by the sum, \( P \), invested initially at interest rate, \( i \), for \( n \) years, gives the accumulated future sum, \( F \). Unless otherwise mentioned, the interest is assumed to be compounded annually.

\[
F = P (1 + i)^n \quad \text{or} \quad F/P = (1 + i)^n
\]

(b) **Present Worth Factor** \((P/F, i\%, n)\) This factor gives the present worth, \( P \), of a future sum, \( F \), discounted at interest rate, \( i \), over \( n \) years.

\[
P = F/(1 + i)^n \quad \text{or} \quad P/F = 1/(1 + i)^n
\]

#### 2. Uniform—Annual Series

(a) **Sinking Fund Factor** \((A/F, i\%, n)\) Assume we buy a machine for Rs 10,000 with a service life of 10 years with no salvage value. After 10 years, to utilize the benefit of service from the equipment, we raise a sinking fund by putting equal amount of money at the end of each year for the service life of the machine. Let the discount rate be \( i \). The annual amount \( A \), which when
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invested in a bank at $i$ interest rate for $n$ years will accumulate to the cost of replacement at the end of the service life of the machine, is the annual payment for the sinking fund.

The sinking fund factor (when multiplied by $F$) gives the amount of money, $A'$, to be invested at the end of each year for $n$ years to yield an amount $F$, at the end of $n$ years. $A'$ is the annual equivalent of sinking fund.

![](alignment)

It can be seen from the cash flow diagram that

$$F = A' + A'(1 + i) + A'(1 + i)^2 + \ldots + A'(1 + i)^{n-1}$$

$$= A' \left[ \frac{(1 + i)^n - 1}{i} \right]$$

The sinking fund factor, denoted by the notation $(A'/F, i\%, n)$ is equal to $A'/F = \frac{i}{(1 + i)^n - 1}$.

In this case, a uniform series of equal annual payments is considered. In all the following formulae, all payments are assumed to be made at the end of the period, unless otherwise mentioned.

(b) Compound Amount Factor $(F/A', i\%, n)$

This factor gives the amount, $F$, that will accumulate at the end of $n$ years, if an amount $A'$ is paid annually at the end of each year for $n$ years.

This is the reciprocal of the sinking fund factor $(A'/F, i\%, n)$. The compound amount factor $(F/A', i\%, n)$ is given by

$$F/A' = \frac{i}{(1 + i)^n - 1}$$

(c) Capital Recovery Factor $(A/P, i\%, n)$

If an amount $P$ is invested in a bank at interest rate $i$ for $n$ years, the Capital Recovery Factor (CRF) gives the annual amount of money, $A$, that can be withdrawn at the end of each year for $n$ years, such that the entire amount of the initial investment is recovered with interest at the end of the $n$th year.

Put in another way, if a loan of $P$ is taken from a bank at $i$ interest rate, the Capital Recovery Factor computes the annual amount of money, $A$, to be paid at the end of each year for $n$ years, such that the initial loan of $P$ is repaid with interest at the end of the $n$th year.

![](alignment)

Referring to the cash flow diagram, we can write

$$P = \frac{A}{(1+i)} + \frac{A}{(1+i)^2} + \frac{A}{(1+i)^3} + \ldots + \frac{A}{(1+i)^n}$$
Economic Considerations in Water Resources Systems

= \frac{A[(1+i)^n - 1]}{i(1+i)^n},

Therefore,

\frac{A}{P} = \left[ (1 + i)^n - 1 \right] i(1 + i)^n

Note that both A and A' refer to annual values. The notation A is used with reference to the present amount P, and A' with reference to the future money F, to express the fact that A and A' are different in value.

\textbf{(d) Present Worth Factor \( P/A, i\%, n \)} This factor gives the present worth of equal annual payments made at the end of each year for n years if the interest rate is i.

This is the reciprocal of the Capital Recovery Factor and is equal to

\frac{P}{A} = \left[ (1 + i)^n - 1 \right] i(1 + i)^n

It may be noted that the Capital Recovery Factor, \( \frac{A}{P}, i\%, n \), accounts for both the interest and the sinking fund due to depreciation and is equal to the sum of the interest rate and the sinking fund factor, as

\left( \frac{A}{P}, i\%, n \right) = i + \frac{i}{\left( 1 + i \right)^n - 1}

This means that the capital recovery is equal to the interest plus depreciation on an annual basis. The equivalence may be seen from the following example.

\textbf{Note:} As \( n \to \infty \), \( \frac{A}{P} \to i \), meaning that over a large repayment period, the capital recovery boils down to the interest with negligible amount to be raised as the sinking fund.

\lim_{n \to \infty} \left( \frac{A}{P}, i\%, n \right) = i

If A is the constant annual payment to be made on a perpetual basis, the equivalent present cost indicated by the value, \( \frac{A}{i} \), is termed as the capitalized cost.

In the examples illustrated in this section, the numerical values of the discount factors, where necessary, were directly used as obtained from the appropriate expressions, for the required values of the interest rate, i, and the number of years, n.

\textbf{Example 3.1.2} Suppose we buy a pump for Rs 10,000 today. It has a service life of 10 years with no salvage value at the end of its service life. We take a loan of Rs 10,000 from the bank at 6% interest rate.

The amount required to be paid at the end of each year to the bank to repay the loan completely in 10 years with interest is

\[ A = 10,000 \times \text{Capital Recovery Factor, } \left( \frac{A}{P}, 6\%, 10 \right) \]

with \( i = 6 \) and \( n = 10 \).

\[ = 10,000 \times 0.13587 = Rs 1,358.7 \]
The loan of Rs 10,000 will be repaid with interest if Rs 1,358.7 is paid to the bank at the end of each year for 10 years.

On the other hand, we bought the pump for Rs 10,000 (borrowed from the bank) and it is in our hands today (but we have no other money). Every year, we raise a fund and, at the end of the year, deposit in the bank to have enough money at the end of 10 years to buy another new pump. Assuming that the cost of the pump remains the same, and that money grows at the same interest rate $i$ (as when we borrowed the money to buy the pump), the annual sinking fund is

\[
A' = 10,000 \times \text{sinking fund factor} \left[ \frac{A}{P}, 6\%, 10 \right] \quad \text{with } i = 6 \text{ and } n = 10
\]

\[
= 10,000 \times 0.07587 = \text{Rs 758.7 (This is the monetary equivalent of the annual depreciation of the pump)}.
\]

With this annual payment, we will have Rs 10,000 at the end of 10 years in the bank to buy a new pump. Only then, at the end of 10 years we will be in the same position as we are now. But remember, we should also pay the bank interest annually on the borrowed money of Rs 10,000 at 6% per year. This annual interest amount is

\[
\text{Annual interest} = P\times i = 10,000 \times 0.06 = \text{Rs 600}
\]

which is the exact difference between the amounts $A$ and $A'$,

\[
(A - A') = 600,
\]

i.e. Amount of capital recovery = Interest + Sinking fund (annual)

### 3.1.3 Amortization

Amortization is the term generally used to indicate payment of a debt in equal instalments at uniform intervals of time. Part of each payment is credited as interest and the remaining towards repayment of loan. For example, if we borrow Rs 100,000 from the bank to be repaid in 10 years at 6 per cent interest rate, then the equal instalment of money to be paid at the end of each year is

\[
A = \text{Annual instalment} = \text{Principal amount} \times \text{Capital recovery factor}
\]

\[
= P \left( \frac{A}{P}, 6\%, 10 \right) = 100,000 \times (\frac{A}{P}, 6\%, 10) = 100,000 \times (0.13587) = 13,587
\]

This is the exact amount to be paid at the end of each year. For any given year, this has two components, one is the interest on the outstanding balance of the loan at the beginning of the year and the other is the amount credited towards repayment of the loan itself.

Each year, though we pay to the bank the same amount of Rs 13,587 annually, the amount credited towards interest and towards debt reduction (repayment) will be different in different years, because the interest in a year is computed based on the outstanding balance (of debt) at the beginning of the year. For example, for the first year, the interest component will be 6% of 100,000, which is 6,000. At the end of first year, then, out of Rs 13,587 paid to the bank, the bank would credit Rs 6,000 as interest, and the remaining amount of Rs 7,587 ($= \text{Rs 13,587} - \text{Rs 6,000}$) towards repayment of the loan. Thus the outstanding loan at the beginning of the second year is Rs 100,000 – Rs 7,587 = Rs 92,413, and so on.
We can determine the components of interest and repayment (of loan) out of the equal annual instalment paid, at the end of any year, \( x \), as follows:

**Repayment**

Repayment is made by building a sinking fund through equal annual payments, \( A \), at the end of each year. The total fund in \( n \) years at \( i\% \) interest rate should accumulate to \( P \), when an amount \( A \) is deposited in the bank at the end of each year for \( n \) years.

Thus \( A' = (\text{Principal borrowed}) \times (\text{Sinking fund factor}) = P \left( A'/F, i\%, n \right) \)

The total amount paid towards debt or loan at any point of time is equal in value to the sinking fund raised till that point of time.

\[
D_x = A' + A'(1+i) + A'(1+i)^2 + \ldots + A'(1+i)^{n-1} = A' \left[ (1+i)^n - 1 \right]/i
\]

Similarly, the total amount paid towards repaying the loan till the end of year \( x - 1 \) is

\[
D_{x-1} = A' \left[ (1+i)^{x-1} - 1 \right]/i
\]

Therefore, the amount paid towards repayment of loan in the year \( x \),

\[
\Delta D_x = D_x - D_{x-1} = A' \left[ (1+i)^x - 1 \right]/i - A' \left[ (1+i)^{x-1} - 1 \right]/i
\]

We know that \( A' \), the annual sinking fund, is equal to

\[
A' = P \left( A'/F, i\%, n \right) = P \left[ i/(1+i)^n - 1 \right]
\]

Therefore, \( \Delta D_x = P(i(1+i)^{x-1} - (1+i)^x - 1) \),

which is the repayment component in the year \( x \). Since \( A = \text{Interest} + \text{Repayment} \), the interest component for year \( x = A - \Delta D_x \).

**Example 3.1.3** For the problem mentioned earlier, where \( P = 100,000 \), \( i = 6\% \) and \( n = 10 \) yrs, the yearly components of repayment and interest are shown in the following table.

<table>
<thead>
<tr>
<th>Year</th>
<th>Repayment, ( \Delta D_x )</th>
<th>Balance ( A' )</th>
<th>Interest</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7,386.80</td>
<td>92,413.20</td>
<td>6,000.00</td>
</tr>
<tr>
<td>2</td>
<td>8,042.00</td>
<td>84,371.20</td>
<td>5,544.79</td>
</tr>
<tr>
<td>3</td>
<td>8,524.52</td>
<td>75,846.68</td>
<td>5,062.27</td>
</tr>
<tr>
<td>4</td>
<td>9,036.00</td>
<td>66,810.68</td>
<td>4,550.80</td>
</tr>
</tbody>
</table>

(Contd.)
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(Contd.)

\[ P = 100,000; \ i = 6\%, \ n = 10 \text{ yrs} \]

\[ A = 13,586.80 \quad A' = 7,586.80 \]

<table>
<thead>
<tr>
<th>Year</th>
<th>Repayment, ( \Delta D_i )</th>
<th>Balance</th>
<th>Interest</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>9,578.15</td>
<td>57,232.53</td>
<td>4,008.64</td>
</tr>
<tr>
<td>6</td>
<td>10,152.84</td>
<td>47,079.68</td>
<td>3,433.95</td>
</tr>
<tr>
<td>7</td>
<td>10,762.01</td>
<td>36,317.67</td>
<td>2,824.78</td>
</tr>
<tr>
<td>8</td>
<td>11,407.74</td>
<td>24,909.93</td>
<td>2,179.06</td>
</tr>
<tr>
<td>9</td>
<td>12,092.20</td>
<td>12,817.73</td>
<td>1,494.60</td>
</tr>
<tr>
<td>10</td>
<td>12,817.73</td>
<td>0</td>
<td>769.06</td>
</tr>
</tbody>
</table>

col 1: year number,
col 2: repayment in the year, \( \Delta D_i \) (end of year value),
col 3: outstanding balance of loan at the beginning of the year in col 1 \( = P - D_i \),
col 4: Interest for the year credited at the end of the year \( = A - \Delta D_i \)

Note that the interest component will be relatively high and the repayment low in the initial years, while the trend reverses in the latter years.

Example 3.1.4 A person borrows a sum of Rs 50,000 @ 6 per cent interest for 10 years. Determine the equal annual sum to be paid at the end of each year to repay the loan, and the amounts credited towards interest and repayment in the 5th year.

1. The amount of money to be paid at the end of each year
   \( = 50,000 \left\{ \frac{P}{A}, \ 6\%, \ 10 \right\} = 50,000 \ . \left(0.1359 \right) = 6795 \)
2. Year \( x = 5 \); \( A = 6795 \)
   Annual sum paid at the end of 5th year, \( A = Rs \ 6795 \)
   The sinking fund, \( A' = 50,000 \left( \frac{A'}{P}, \ 6\%, \ 10 \right) \)
   \( = 50,000 \cdot \left(0.0759 \right) = 3795 \)
   The repayment component (reduction in the principal borrowed)
   in the 5th year \( = A' \left( 1 + i \right)^{-1} \)
   Payment, \( \Delta D_5 = 3795 \left( 1 + 0.06 \right)^{-1} = 4791 \)
   \( \therefore \) Interest component \( = A - \Delta D_5 \)
   \( = 6795 - 4791 = 2004 \)
   \( \therefore \) At the end of 5th year,
   interest component \( = Rs \ 2004 \)
   repayment component \( = Rs \ 4791 \)
   out of the annual amount of Rs 6795 paid

Annual Costs The annual cost (AC) of a project consists of the following:
1. Interest on the borrowed capital
2. Depreciation or amortization cost
3. Operation and maintenance cost (O&M cost), and
4. Other costs, specified, if any
   The first two costs sum up to the capital recovery on an annual basis.
The background information dealt with so far will be useful in the economic evaluation of alternative proposals for specified needs. The comparison may be based on any of several available methods, but a proper economic evaluation (based on any method) should result in an identical selection. Two of the most common methods used are illustrated in the following pages: present worth method and the equivalent annual cost method. The use of these two methods is illustrated next with examples.

Note that the present worth is a function of the duration of analysis and, therefore, any comparison of alternatives should be based on the same period of analysis for each alternative. In this respect, comparison on the basis of equivalent annual cost is relatively easier.

Example 3.1.5 Which of the following plans is more economical at 6% interest rate?

<table>
<thead>
<tr>
<th></th>
<th>Plan A</th>
<th>Plan B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost of Equipment</td>
<td>50,000</td>
<td>35,000</td>
</tr>
<tr>
<td>Annual O &amp; M Costs</td>
<td>2,000</td>
<td>2,500</td>
</tr>
<tr>
<td>Salvage value</td>
<td>7,000</td>
<td>6,000</td>
</tr>
<tr>
<td>Service life</td>
<td>30 years</td>
<td>15 years</td>
</tr>
</tbody>
</table>

Let us compare the two alternate plans by different methods

1. Equivalent Annual Cost (AC)

**Plan A**

Annual Cost = Interest + Depreciation + O & M costs

Interest on borrowed capital = 50,000 × .06 = 3000

Depreciation = \((P - L) \frac{A}{F}, 6\%, 30\) = \((50,000 - 7,000) (.01265) = 543.95\)

O & M Costs = 2000

Total AC = 3000 + 543.95 + 2000 = 5543.95

**Plan B**

Interest = 35,000 × .06 = 2100

Depreciation = \((35,000 - 6,000) \frac{A}{F}, 6\%, 15\) = 29,000 × (0.04296) = 1245.84

O & M Cost = 2500

Total AC = 2100 + 1245.84 + 2500 = 5845.84

Plan A is preferable because of lower annual cost.

2. Present Worth (PW) Comparison

As mentioned earlier, PW should be compared for equal periods of analysis in both plans. Let us compare the plans on a 30-year basis. For this purpose, the life of plan B is extended by another 15 years with
identical expenditure pattern as in the first 15 years. A comparison of present worth of all costs can be made then, based on an analysis period of 30 years.

**Plan A** (30 years)

<table>
<thead>
<tr>
<th>Cost Component</th>
<th>Present Worth Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Cost</td>
<td>50,000</td>
</tr>
<tr>
<td>Salvage Value</td>
<td>–7000 (P/F, 6%, 30)</td>
</tr>
<tr>
<td>(negative cost)</td>
<td>–7000 (0.17411) = –1218.77</td>
</tr>
<tr>
<td>O &amp; M Costs</td>
<td>2000 (P/A, 6%, 30)</td>
</tr>
<tr>
<td></td>
<td>2000 (13.7648) = 27,529.60</td>
</tr>
<tr>
<td></td>
<td>76,310.83</td>
</tr>
</tbody>
</table>

**Plan B** (for 30 years)

<table>
<thead>
<tr>
<th>Cost Component</th>
<th>Present Worth Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Cost</td>
<td>35,000</td>
</tr>
<tr>
<td>35,000 at 15 years</td>
<td>35,000 (P/F, 6%, 15)</td>
</tr>
<tr>
<td></td>
<td>35,000 (.41727) = 14604.45</td>
</tr>
<tr>
<td>6000 (negative cost at 15 years)</td>
<td>–6000 (P/F, 6%, 15)</td>
</tr>
<tr>
<td></td>
<td>–6000 (.41727) = –2503.62</td>
</tr>
<tr>
<td>6000 (negative cost at 30 years)</td>
<td>–6000 (P/F, 6%, 30)</td>
</tr>
<tr>
<td></td>
<td>–6000 (17411) = –1044.66</td>
</tr>
<tr>
<td>O &amp; M Costs</td>
<td>2500 (P/A, 6%, 30)</td>
</tr>
<tr>
<td></td>
<td>2500 (13.7648) = 34,412</td>
</tr>
<tr>
<td>Total Costs</td>
<td>35,000 + 14,604.45 – 2503.62 – 1044.66 + 34,412</td>
</tr>
<tr>
<td></td>
<td>80467.72</td>
</tr>
</tbody>
</table>

Because of lower costs Plan A is preferred to Plan B.

**Note:** The annual costs worked out in method 1 may be obtained from the present worth calculated above by applying the capital recovery factor, \((A/P, 6\%, 30) = 0.07265\).

\[ AC_{A} = (PW_{A}) \times (A/P, 6\%, 30) = (76,310.83) (0.07265) = 5543.98 \]

\[ AC_{B} = (PW_{B}) \times (A/P, 6\%, 30) = (80,465.72) (0.07265) = 5845.98 \]

which are same as before except for rounding off of errors.

**Example 31.9** There are two alternatives to water supply in an irrigation district. Plan A is to construct an open channel at an initial cost of Rs 50,00,000 with O & M cost of Rs 40,00,000 per year. The alternative, Plan B, is to go for a piping system at an initial cost of Rs 90,00,000 and an O &
M cost of Rs 50,000/year. Money is available for 6% interest rate, and sinking fund will improve at 4% interest. Useful life of the project in both cases is 20 years. Select the more economical of the two alternatives.

Let us work out the annual costs in both cases.

<table>
<thead>
<tr>
<th>Case</th>
<th>Plan A</th>
<th>Plan B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest</td>
<td>300,000</td>
<td>540,000</td>
</tr>
<tr>
<td>Depreciation</td>
<td>167,500</td>
<td>301,500</td>
</tr>
<tr>
<td>$(A/F, 4%, 20) = 0.0335$ on $P$</td>
<td>867,500</td>
<td>891,000</td>
</tr>
<tr>
<td>O &amp; M Costs</td>
<td>400,000</td>
<td>50,000</td>
</tr>
</tbody>
</table>

\[ \therefore \text{Plan A is preferable to Plan B.} \]

Note: In this example, the interest rate on borrowed money and the discount rate for depreciation are different.

### Problems

3.1.1 A bank gave 10 per cent interest, compounded every two months for the first six months of a year. Subsequently, the bank decided to give only 8 per cent interest, compounded monthly, for the rest of the year. What was the effective rate of interest for that year? (Ans: 9.36%)

3.1.2 An equipment costs Rs 800,000 today and has a service life of 15 years. If the salvage value at the end of 15 years is Rs. 200,000, determine its appraisal value at the end of 10 years. Interest rate = 10%. (Ans: Rs 499,070)

3.1.3 A person takes a loan from a bank: Rs 5000 now, Rs 2000 at the end of 2 years, Rs 3000 at the beginning of the 5th year, and equal amounts of Rs 1000 each beginning 6th year for 10 years. Determine what equal annual instalments (end of year) he should pay for 15 years in order to repay the loan with interest at 10 per cent. (Ans: Rs 1696.19)

3.1.4 A pipeline is proposed to be laid from an existing pumping station to a nearby reservoir. Two alternate possible pipe sizes are considered:

<table>
<thead>
<tr>
<th>Pipe size</th>
<th>Cost/hr of pumping (Rs)</th>
<th>First cost of construction (Rs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5.0</td>
<td>80,000</td>
</tr>
<tr>
<td>B</td>
<td>3.0</td>
<td>160,000</td>
</tr>
</tbody>
</table>

Both pipe sizes have a life of 15 years, with no salvage value. Interest rate is 6%.

(i) Which is the most economical pipe size, if the total number of hours of pumping per year is 5000. Compare equivalent annual costs.

(ii) How many hours of pumping per year are required to make the two pipe sizes equally economical. (Ans: (i) pipe size A (ii) 4120 hrs/yr)
3.2 ECONOMIC ANALYSIS

A discussion of the market supply and demand is necessary before we go into the benefit cost analysis of project planning.

Types of Goods in the Market In general, goods can be classified into two categories:

1. Normal goods, or personal goods, or private goods, and
2. Collective goods, or public goods.

Normal goods have the property of being consumed by individuals, such that, given a fixed quantity, more units for one means fewer for others and vice versa. Most of the goods for which a real market exists come under this category. For example, for a fixed number of automobiles produced, the more one section of the people buy them, the less they are available for others (to buy), and vice versa. However, there are some other goods for which this is not true. National defence and flood control are examples of these. Here each one “consumes” the same amount of national defence, and, although each person added to the population consumes national defence, his/her consumption does not necessarily make it less available for the rest of the population. These goods are called collective goods, or public goods, in contrast to the normal, or personal or private goods.

3.2.1 Market Demand and Supply

Demand is the quantity per unit time that people will be willing to buy, expressed as a function of the price, all other factors remaining constant. The demand curve has the property that it slopes downward to the right because lower prices increase sales (more number of buyers), and higher prices decrease them (less number of buyers).

Price Elasticity of Demand

Let us consider a demand curve, Fig. 3.1, which depicts a linear relation between price and the quantity demanded.

A demand curve indicates the change in the demand or sales for a given change in the price. This relationship is given by the price elasticity of demand,
Economic Considerations in Water Resources Systems

E, which is defined as the ratio of the relative change in the quantity, for a given relative change in the price. If \( Q \) is the demand at price \( P \), then the price elasticity of demand \( E \) is given by

\[
E = -\left( \frac{\Delta Q/Q}{\Delta P/P} \right) = -\left( \frac{\Delta Q}{\Delta P} \right) \left( \frac{P}{Q} \right) = -\frac{Q/P}{\Delta P/\Delta Q} \quad (12.1)
\]

The negative sign indicates that \( Q \) increases as \( P \) decreases (slope of the demand curve is negative). In the case of the linear demand curve shown in Fig. 3.1, though the slope remains constant at all points on it, the elasticity of demand changes from point to point on the curve. It varies from infinity along the vertical axis \( (E = \infty) \) to zero along the horizontal axis \( (E = 0) \).

A value for \( E \) of infinity (point \( A \) on the linear demand curve) indicates a perfectly elastic product, which no one will buy if the price is raised. At this price, goods become perfectly elastic and are completely priced out of the market. A value for \( E \) of zero (point \( B \) on the linear demand curve) indicates a perfectly inelastic product, i.e. one for which the price has no effect on demand.

In between these two extremes, a given product may be elastic or inelastic at a given price, depending on its relation to the demand as defined by the demand curve, in other words, on the price elasticity of demand, \( E \).

As the price is reduced from point \( A \), elasticity decreases, until it reaches unity at some point \( C \), when the product is no longer said to be elastic. For all points on the line above \( C \), the price elasticity of demand, or simply the elasticity as it is often called, is \( E > 1 \). The total revenue, which is the product of the price and the quantity of goods sold \( (PQ) \), increases up to this point, as the increase in sales (demand) offsets the reduction in price resulting in an increase in the gross revenue. The point \( C \) (point of unit elasticity, \( E = 1 \)) provides the supplier the largest revenue. If the price is reduced further down the point \( C \), though demand continues to increase, the increase will not be fast enough to offset the decreasing price resulting in a decline in the gross revenue from that point on. For all points below \( C \), the price elasticity will be less than unity, \( (E < 1) \). In this region, the product is said to be inelastic. It is to be noted that the same product is inelastic at low prices and elastic at high prices.

**Supply Curve**

The supply curve shows the relationship between the quantity produced and the price at which the producers are willing to sell, other things remaining constant. Typically, a supply curve slopes upward to the right, meaning that more goods will be produced, and more sellers will enter the market as the price increases.

**Market Price Determination**

A market consists of sellers and buyers. The behavior of each group is, however, different. The buyers’ behavior is reflected by the demand curve and that of the sellers by the supply curve. The buyers would like to minimize the expenditure and the sellers would like to maximize their revenue from sales.
The demand curve and the supply curve therefore combine to establish the equilibrium market price. The equilibrium price is the minimum under conditions of pure competition that each individual buyer must pay per unit quantity purchased and the maximum that each seller can receive for each unit quantity sold (James and Lee, 1971).

That this equilibrium point corresponds to a point of unit elasticity on the demand curve can be shown as follows:

The total revenue from sales or the total expenditure, $G$, for goods purchased is given by

$$G = \text{price} \times \text{quantity} = P \cdot Q$$

For this to be optimum, $dG/dP = 0$

$$dG/dP = P \cdot dQ/dP + Q = 0$$

or

$$dQ/dP = -Q/P$$

or

$$- (P/Q) \cdot dQ/dP = E = 1$$

The left-hand side is the price elasticity of demand by definition. Therefore, the market equilibrium prevails at the price corresponding to $E = 1$ on the demand curve. This analysis presumes that free market conditions exist with no restrictions and there is free entry for an unlimited number of buyers and sellers. At this equilibrium price, the sellers would have maximized their returns and the buyers would have minimized their total expenditure.

We will not discuss in this book the consequences of a shift in demand or the effect of subsidies on market behavior.

### 3.2.2 Aggregation of Demand

The procedure for aggregating demand curves depends on the type of product. In the case of a normal product (market good), where consumption is mutually exclusive (what is consumed by one is not available to the others—e.g. drinking water), the combined demand curve is generated by adding the demands of each individual at a given price. Thus, the combined demand is computed at each price and plotted to get the aggregate demand curve. This horizontal addition of demands is characteristic of normal or market goods.

In the case of a collective good, however, where the consumption is not mutually exclusive (example—flood control), the prices are added up for a given demand and for all demand levels. This type of vertical addition of price is characteristic of collective or public goods.

Figures 3.2 and 3.3 show examples of how this aggregation is done graphically for normal goods (such as irrigation water) and for collective goods (such as flood control), respectively.

The aggregate demand curves can be arrived at analytically as well, if the equations of individual demand curves are given.
Water is supplied from a project for two types of uses: rural and urban. The demand curve for rural use is given by \( P + 3y = 30 \), and the demand curve for urban use is \( 4P + y = 40 \), where \( y \) is the demand and \( P \) is price in appropriate units. Determine the combined demand curve.

**Graphical Method**

The rural and urban demand curves are first plotted individually. The combined demand curve is obtained by adding the individual demands at a given price as, in this case, water is a normal good.
Let rural demand = $y_r$ at price $P$
urban demand = $y_u$ at price $P$
y = $y_r + y_u$ at the same price $P$ for a normal good.
y_r = (30 – $P$)/3
$y_u = 40 – 4P$

For $P \leq 10$ (or $y \geq 20/3$),
y = $y_r + y_u = (30 – P)/3 + 40 – 4P = 50 – (13/3)P$, or $13P + 3y = 150$,
where $y$ is the combined demand.

For $P \geq 10$, (or $y \leq 20/3$),
y = $y_r = (30 – P)/3$ or $P + 3y = 30$

Combined demand curve is given by

P + 3y = 30 for $P \geq 10$

and

13P + 3y = 150 for $P \leq 10$.

Two different groups of people are affected by a flood control project. The demand for flood control from the two groups are as follows:

Group 1 $P + 3y = 30$
Group 2 $4P + y = 40$
where $P$ is price and $y$ is the level of protection provided.

Determine the aggregate demand curve.

**Graphical Method** In this case, the "consumption" is not mutually exclusive as flood control is a "collective good". Therefore, the combined demand is
obtained by adding the prices the groups are willing to pay for the same level of protection, vertically.
Let \( P_1 \) = price for group 1 at demand level \( y \).
Let \( P_2 \) = price for group 2 at demand level \( y \).
\[ P_1 = 30 - 3y \]
\[ P_2 = (40 - y)/4. \]
For \( y \leq 10 \), \( P = P_1 + P_2 = (30 - 3y) + (40 - y)/4 \), or \( 4P + 13y = 160 \).
For \( y \geq 10 \), \( P = P_1 = 30 - 3y \) or \( P + 3y = 30 \).

Problems

3.2.1 A water project is proposed to supply water for municipal and irrigation uses. Municipal demand is given by \( P + 2Y = 10 \), and irrigation demand is given by \( 2P + Y = 20 \), where \( P \) is the price and \( Y \) is the demand.
(i) Determine the aggregate demand curve.
(ii) Assuming the total cost curve is given by \( C = \sum x_i Y_i \), determine the optimal level of \( Y \).
(iii) Determine the share of municipal and irrigation supplies at optimal level of \( Y \).
(Ams: (ii) \( Y = 10 \), (iii) \( Y \) (municipal) = 2, \( Y \) (irrigation) = 8)

3.2.2 Consider the combined demand curve for rural and urban users obtained in Example 3.2.1. If the combined demand with and without the water project is 5 and 20 respectively, estimate the benefits from the project.
(Ams: 133.65)

3.3 CONDITIONS OF PROJECT OPTIMALITY

3.3.1 Water Resources as a Production Process

Water resources development is a production process. The basic purpose of production is to convert a set of inputs to a set of outputs. Examples of output for a water resources project are irrigation, hydropower generation and flood damage alleviation. Examples of inputs are natural streamflow, cement, concrete, steel and turbines.

Let the elements of the input vector, \( X \), consist of individual inputs, \( x_1, x_2, \ldots, x_m \). Similarly, the output vector, \( Y \), consists of individual outputs, \( y_1, y_2, y_3, \ldots, y_n \).

The Production Function

The production function defines the relationship between the set of outputs that can be produced from a set of inputs, reflecting, in a way, the efficiency of the production process itself. The following is an example of a production function for two outputs, say irrigation water and hydropower generation from a reservoir, in which power is produced from the water released downstream of it (in a riverbed powerhouse).
The region below the curve containing the origin is the technologically feasible region. Any combination of outputs above the curve (on the side of the curve away from the origin) is infeasible. A combination of outputs represented by points within the feasible region will be inefficient. For example, for a given amount of irrigation water supply, \( y_1 \), there is a maximum of hydropower, \( y_2 \), that can be produced from the reservoir, represented by the point \( A \) on the curve. Anything below \( A \) is feasible, but inefficient, and anything above \( A \) is infeasible. Thus \( A \) represents the maximum possible \( y_2 \) corresponding to \( y_1 \). Thus, the curve represents the locus of efficient points in the production process. This is also called the efficiency frontier curve or production possibility frontier. The production function is a mathematical representation of this line. It is related to the input and output vectors and, putting all the terms on the left-hand side, it is expressed as

\[
 f(X, Y) = 0 \quad (3.1)
\]

While every point on the production function represents an efficient point (efficient operation of the system), the selection of the best point, however, requires value judgment. Each point on the production function (also termed a pareto-admissible solution) represents a relative value associated with each variable. The objective function is a function of both the input and the output vectors. The net benefit objective function, \( u = u(X, Y) \), is expressed as

\[
 u = \sum_i b_i y_i - \sum_j c_j x_j \quad (3.2)
\]

where \( b_i \) refers to the unit benefit associated with the \( i \)th output element, and \( c_j \) refers to the unit cost associated with the \( j \)th input element.

**Objective Function**

The objective is to maximize net benefits such that the solution point lies on the production frontier line, i.e. to maximize the objective function, \( u = u(X, Y) \), subject to the constraint, \( f(X, Y) = 0 \), where \( X \) and \( Y \) are input and output vectors containing \( m \) and \( n \) elements, respectively.

The Lagrangean formed from the objective and the constraint is written as

\[
 L = u(X, Y) - \lambda f(X, Y) \quad (3.3)
\]
where $\lambda$ is the Lagrange multiplier and $L$ is the function to be maximized. The necessary conditions for optimality are obtained by differentiating $L$ partially with respect to each $x_i$, each $y_j$, and setting each partial differential to zero. Thus, there will be $m + n + 1$ equations with $m + n + 1$ unknowns ($m$ inputs, $n$ outputs, and $\lambda$). Solving this set of simultaneous equations results in the optimal levels of inputs and outputs.

$$\frac{\partial L}{\partial x_i} = \lambda \frac{\partial f}{\partial x_i} \quad \text{for } i = 1, 2, \ldots, m \quad (3.4)$$

$$\frac{\partial L}{\partial y_j} = \lambda \frac{\partial f}{\partial y_j} \quad \text{for } j = 1, 2, \ldots, n \quad (3.5)$$

Setting this last equation to zero means that the constraint, Eq. (3.1) must be satisfied.

By dividing Eq. (3.4) for two inputs $x_1$ and $x_2$, Eq. (3.5) for two outputs $y_1$ and $y_2$, and pairs of equations for one input, $x_i$, and one output, $y_j$, one obtains the following equations:

$$\frac{\partial x_i}{\partial x_1} \frac{\partial x_1}{\partial y_1} \frac{\partial x_2}{\partial y_2} \quad (3.6)$$

$$\frac{\partial x_i}{\partial x_1} \frac{\partial x_1}{\partial y_1} \frac{\partial x_2}{\partial y_2} \quad (3.7)$$

$$\frac{\partial x_i}{\partial x_1} \frac{\partial x_1}{\partial y_1} \frac{\partial x_2}{\partial y_2} \quad (3.8)$$

Since $f(x, y)$ must equal zero, an increase in one element must be offset by a decrease in another. Thus:

$$\frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial y_1} \frac{\partial x_2}{\partial y_2} \quad (3.9)$$

$$\frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial y_1} \frac{\partial x_2}{\partial y_2} \quad (3.10)$$

$$\frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial y_1} \frac{\partial x_2}{\partial y_2} \quad (3.11)$$

Combining Eqs. (3.6) and (3.9), (3.7) and (3.10), and (3.8) and (3.11), one finally gets

$$\frac{\partial x_i}{\partial x_1} \frac{\partial x_1}{\partial y_1} \frac{\partial x_2}{\partial y_2} \quad (3.12)$$

$$\frac{\partial x_i}{\partial x_1} \frac{\partial x_1}{\partial y_1} \frac{\partial x_2}{\partial y_2} \quad (3.13)$$

$$\frac{\partial x_i}{\partial x_1} \frac{\partial x_1}{\partial y_1} \frac{\partial x_2}{\partial y_2} \quad (3.14)$$

### 3.3.2 Conditions of Optimality

The following can be interpreted easily.

$$\frac{\partial x_i}{\partial x_1} = \text{rate of change of net benefit function with respect to input } i,$$

or the marginal cost of input $x_i = MC_i$

$$\frac{\partial x_i}{\partial y_1} = \text{rate of change of benefit function with respect to output } j,$$

or the marginal benefit of output $y_j = MB_j$

It is to be emphasized that, whenever the change of one element with respect to another is considered, all other elements are held at constant levels.

The right-hand side of Eq. (3.12) is termed as the marginal rate of substitution, $MRS_{x_1}$, which is defined as the marginal rate at which the second input needs to be substituted for the first input, holding the level of production constant. Thus, Eq. (3.12) gives

$$MC_i/MC_j = MRS_{x_1} \quad (3.15)$$
The right-hand side of Eq. (3.13) is termed as the **marginal rate of transformation**, $MRT_{21}$. Thus Eq. (3.13) gives

$$\frac{MB_1}{MB_2} = MRT_{21} \quad (3.16)$$

The right-hand side of Eq (3.14) is termed as the **marginal physical productivity** of the $i$th input when devoted to the $j$th output, or the **marginal physical product**, $MPP_{ij}$. Thus Eq (3.14) gives

$$\frac{MC_i}{MB_j} = MPP_{ij} \quad (3.17)$$

Equations (3.15), (3.16), and (3.17) are the three conditions of optimality that are necessary to achieve maximum net benefit.

**Note:** It must be noted that the conditions of optimality are necessary but not sufficient conditions. Sufficiency requires that there be no higher value of the objective function than is indicated by the solution. Even then, the solution can only be a local maximum and need not be a global maximum. For a more practical interpretation of those theoretically derived expressions, see Maass et al. (1968) and James and Lee (1971).

**Example 3.5.1** Combination of Inputs (Marginal Rate of Substitution)

A water resources project has the option of using two inputs, reservoir water and groundwater, to provide irrigation to a given area. All other things being equal, the need is satisfied by the combination of inputs, $x_1$ and $x_2$, as shown in Fig. 3.5. To determine the best combination of these inputs, the first optimality condition says that the levels of $x_1$ and $x_2$ should be in the ratio of $MC_2$ to $MC_1$.

![Fig. 3.5 Combination of Inputs](image)

**Table 3.5.1**

<table>
<thead>
<tr>
<th>Sl No.</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\Delta x_1$</th>
<th>$\Delta x_2$</th>
<th>$MRS_{21}$</th>
<th>Total Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>80</td>
<td>10</td>
<td>15</td>
<td>15</td>
<td>1</td>
<td>750</td>
</tr>
<tr>
<td>2</td>
<td>65</td>
<td>25</td>
<td>10</td>
<td>15</td>
<td>1.5</td>
<td>710</td>
</tr>
<tr>
<td>3</td>
<td>55</td>
<td>40</td>
<td>15</td>
<td>15</td>
<td>1.5</td>
<td>710</td>
</tr>
<tr>
<td>4</td>
<td>45</td>
<td>65</td>
<td>25</td>
<td>10</td>
<td>2.5</td>
<td>710</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>190</td>
<td>125</td>
<td>5</td>
<td>5</td>
<td>960</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>340</td>
<td>150</td>
<td>10</td>
<td>10</td>
<td>1410</td>
</tr>
</tbody>
</table>

$MC_1 = 10; MC_2 = 4; \ (MRS_{21})_{opt} = \frac{MC_2}{MC_1} = 2.5$
For the data shown in the table, the optimal marginal rate of substitution is \( MC_1/MC_2 = 2.5 \). This would correspond to \( x_1 = 50 \) and \( x_2 = 45 \), as shown in the figure. At this point, it is seen that a line with a slope of \( -\frac{\partial x_2}{\partial x_1} = 2.5 \left( \frac{MC_1}{MC_2} \right) \) is tangential to the curve as desired. The total cost would be a minimum (= 680, in the present example) at this point.

### Example 3.1.2 Combination of Inputs (Marginal Rate of Substitution)

Two inputs \( x_1 \) and \( x_2 \) should be combined to satisfy the relationship \( x_1 x_2 = 16 \), in order to produce constant results (output), all other things being the same. Example: \( x_1 \) and \( x_2 \), for example, may represent the quantum of surface water and groundwater to provide a given level of irrigation to a given area. If the marginal costs of \( x_1 \) and \( x_2 \) are 4 and 1, respectively, determine the optimum values of \( x_1 \) and \( x_2 \).

The desired point \((x_1, x_2)\) should lie on the curve \( x_1 x_2 = 16 \). From the first condition of optimality for two inputs,

\[
\frac{\partial x_1}{\partial x_2} = \frac{MC_1}{MC_2} = 4
\]

where \( x_1 x_2 = 16 \); \( \frac{\partial x_1}{\partial x_1} + \frac{\partial x_2}{\partial x_2} = 0 \) or \( \frac{\partial x_1}{\partial x_2} = -\frac{x_2}{x_1} \)

\[
\frac{\partial x_1}{\partial x_2} = \frac{x_2}{x_1} = \frac{MC_1}{MC_2} = 4 \quad \therefore \quad x_2 = 4x_1
\]

As \((x_1, x_2)\) should lie on the curve, \( x_1 (4x_1) = 16 \)

\[
\therefore \quad x_1 = 2 \quad \text{and} \quad x_2 = 4x_1 = 8
\]

The minimum cost at this level of input combination = \( x_1 (MC_1) + x_2 (MC_2) \)

\[
= 2(4) + 8(1) = 16
\]

**Alternate Method**

\[
C = \text{Total cost of production} = x_1 (MC_1) + x_2 (MC_2)
\]

\[
= 4x_1 + 1x_2
\]

For \( C \) to be a minimum, \( \frac{\partial C}{\partial x_1} = 0 \), or \( 4 + \frac{\partial x_1}{\partial x_1} = 0 \)

or

\[
\frac{\partial x_1}{\partial x_1} = -4
\]

The minimum point also lies on the curve \( x_1 x_2 = 16 \), which by differentiation gives

\[
x_1 \frac{dx_2}{dx_1} + x_2 = 0 \quad \text{or} \quad -4 \frac{x_2}{x_1} = -x_2
\]
As \( x_1x_2 = 16, x_1 = 2 \) and \( x_2 = 8 \), as before.

Note: The results are the same in both methods as both tend to maximize the objective of net benefits. But in this case, since the benefits are constant, optimization boils down to minimizing the cost.

**Example 3.3.3 Combination of Outputs (Marginal Rate of Transformation)**

Two outputs can be produced from a production process such that the levels of the two outputs, \( y_1 \) and \( y_2 \), satisfy the relation, \( y_1^2 + 4y_2^2 = 4 \), keeping all other things constant. If the marginal benefits of \( y_1 \) and \( y_2 \), are respectively equal to 10 and 5, determine the optimum levels of \( y_1 \) and \( y_2 \).

\( y_1 \) and \( y_2 \) can denote, for example, irrigated area and hydropower generated respectively. It is assumed that the water used for these two is mutually exclusive. What is used for one is not available for the other. This can result if water is withdrawn from reservoir storage for diversion into canals for irrigation, and hydropower is produced only from the water released downstream of the reservoir.

Optimum value of \( y_1 \) and \( y_2 \) should lie on the curve (production function)

\[
y_1^2 + 4y_2^2 = 4
\]

Diff. with respect to \( y_1 \),

\[
2y_1 + 8y_2 \frac{\partial y_2}{\partial y_1} = 0
\]

or

\[
\frac{\partial y_2}{\partial y_1} = -\frac{1}{4} \frac{y_1}{y_2}
\]

From the second condition of optimality,

\[
\frac{\partial y_2}{\partial y_1} = \frac{MRT_{11}}{MB_1} = \frac{MB_2}{MB_1}, \text{ and } MB_1 = 10, MB_2 = 5
\]

\[
\therefore \quad + \frac{1}{2} \frac{y_1}{y_2} = \frac{10}{5} = 2, \text{ or } y_1 = 8y_2
\]

As \( y_1^2 + 4y_2^2 = 4 \), \( 64y_2^2 + 4y_2^2 = 4 \) or \( y_2 = 1/\sqrt{17} \) and \( y_1 = 8\sqrt{17}/17 \).

The benefit at this level will be maximum

\[
= y_1(MB_1) + y_2(MB_2)
\]

\[
= 8\sqrt{17}(10) + 1/\sqrt{17}(5)
\]

\[
= 85\sqrt{17}
\]

Alternatively, the problem may be solved directly using the Lagrangean multiplier method.

Maximize \[ u = 10y_1 + 5y_2 \]
subject to \[ y_1^2 + 4y_2^2 = 4 \]

**Necessary Condition:**

Lagrangian \[ L = 10y_1 + 5y_2 - \lambda(y_1^2 + 4y_2^2 - 4) \]

\[ \frac{\partial L}{\partial y_1} = 10 - 2\lambda y_1 = 0 \]
\[ \frac{\partial L}{\partial y_2} = 5 - 8\lambda y_2 = 0 \]
\[ \frac{\partial L}{\partial \lambda} = -(y_1^2 + 4y_2^2 - 4) = 0. \]

Solving \[ y_1 = \frac{8}{\sqrt{17}} \]
\[ y_2 = \frac{1}{\sqrt{17}} \]
and \[ \lambda = \frac{85}{\sqrt{17}} \]

[The student may check this for the sufficiency condition.]

**Example 3.1.4 Combination of an Input and an Output (Marginal Physical Product, MPP)**

Consider a reservoir project operated for irrigation and hydropower in which the input element is the water supply, \( x \), and the output element is the hydropower generated, \( y \). Let the marginal cost of water supply be \( MC_x = \alpha \), and the marginal benefit of hydropower generation is \( MB_y = \beta \) where \( \alpha \) and \( \beta \) are constants. Let \( x \) and \( y \) be related as \( y = \sqrt{x} \) within the feasible ranges of \( x \) and \( y \). The question is how much of \( x \) should be used to produce \( y \) under optimal conditions, all other quantities being the same. The rule says:

\[ -\frac{\partial y}{\partial x} = \frac{MPP_y}{MB_y} = \frac{MC_x}{MB_y}, \]

i.e., increase the value of \( x \) until the point where the ratio of marginal output to marginal input equals the marginal physical product, which is the ratio of the marginal input cost to the marginal output benefit.

**Note:** the marginal cost, as determined by partial differentiation of the net benefits, \((\beta y - \alpha x)\), is \(-\alpha\).

When \( y - \sqrt{x} = 0 \), \[ -\frac{\partial y}{\partial x} = -\frac{1}{2\sqrt{x}} = \frac{\alpha}{\beta} \]
\[ x = \frac{1}{4}(\beta \alpha)^2 \]

For maximum net benefits, \( x = 1/4(\beta \alpha)^2 \) and \( y = \frac{1}{2}(\beta \alpha) \)

Alternatively, using the Lagrangean Multiplier method directly.
Maximize \( u = \beta y - \alpha x \)
subject to \( y = \sqrt{x} \equiv 0 \).
\[
L = \beta y - \alpha x - \lambda(y - \sqrt{x})
\]
\[
\frac{\partial L}{\partial x} = -\alpha + \frac{\lambda}{2\sqrt{x}} = 0,
\]
\[
\frac{\partial L}{\partial y} = \beta - \lambda = 0
\]
\[
\frac{\partial L}{\partial \lambda} = -(y - \sqrt{x}) = 0
\]
Solving \( x = \frac{1}{4}(\beta/\alpha)^2, y = \frac{1}{2}(\beta/\alpha) \)
This is a necessary condition for local maximum.
[The student may check this for the sufficiency condition.]

Problems

3.3.1 Water from two different sources is supplied to satisfy the industrial needs of a power plant. Let \( x_1 \) = supply from source 1 at a cost of \( \alpha \)/unit-volume, and \( x_2 \) = supply from source 2 at a cost of \( \beta \)/unit-volume. Any combination of \( x_1 \) and \( x_2 \) such that \( x_1x_2 = k^2 \), where \( k \) is a constant, would satisfy the industrial requirement. Determine the optimal values of \( x_1 \) and \( x_2 \) (for total cost minimization).

3.3.2 The area that can be irrigated, \( x \), and the hydropower that can be generated, \( y \), from a given reservoir are related by the first quadrant of an ellipse, a curve \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = C^2 \), each combination resulting in the best utilization of all inputs, where \( a, b, \) and \( c \) are constants. Determine the optimal levels of \( x \) and \( y \), if the marginal benefits of \( x \) and \( y \) are \( \beta \) and \( \alpha \), respectively.

Note: For a given \( x \), no more of \( y \) than is determined by the equation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = C^2 \) is possible, and vice versa. That is, the optimum \((x, y)\) should lie on this curve and the slope of the tangent at this point should be \((\beta/\alpha)\).

3.4 Benefit Cost Analysis

The primary purpose of a water resources project is its utility to the public. Obviously, a given project will not be beneficial to everyone who may be affected by it. Some will be positively benefitted and some negatively. Benefi-
ciaries can be individuals, communities, groups of people, or entities such as administrative districts. For example, a reservoir may benefit all those downstream by providing the benefits of irrigation, low-flow augmentation, hydropower generation, flood-damage alleviation etc; whereas, those in the upstream suffer from submergence, inconvenience in rehabilitation and resettlement, etc. In this context, a plan will be considered economically feasible only if the positive benefits outweigh the total costs involved in implementing it or, putting it in monetary terms, the benefit cost ratio should be more than one, or the net benefits (gross benefits minus the total costs) should be positive. It is important to note that the benefit cost analysis be made with and without the project rather than before and after. This is because some of the after effects may occur in course of time, even without the project and therefore cannot be counted as a justification for the project. Benefit cost analysis provides an objective assessment of the economic feasibility of each project and provides a means of selecting the best among those short-listed.

3.4.1 Benefits and Costs

Benefit cost analysis requires the estimate of both benefits and costs associated with a plan and should take into account all the parties affected one way or another by it. Benefits accrue through the use of a commodity or service. The value of the use, expressed in monetary terms, is the benefit associated with the plan. There may not exist a market related to the commodity or service and, therefore, estimation of benefits is not always straightforward and easy. Benefit cost analysis requires estimation of both benefits and costs in monetary terms. Benefits are to be estimated by aggregating them to whomsoever they may accrue from the plan. Cost estimates should ideally reflect opportunity costs rather than just the market prices for a true comparison. This is rarely done in practice, however.

3.4.2 Cost and Benefit Curves

The total cost comprises fixed cost and variable cost. Fixed costs, as the term indicates, are fixed and do not depend on the level of project output. Variable costs are marginal costs and vary with level of output. While fixed costs are not marginal, they do affect the computations of the total benefit cost ratio and the decision whether the project should be built at all. For example, in a reservoir sizing problem using LP, if the objective is to minimize the total cost of reservoir construction (sum of fixed and variable costs, variable costs depending on the storage), and if the optimal required storage, works out to zero then, obviously, no reservoir need be built and no costs incurred.

If \( TC = f(Y) \) is the total cost at output \( Y \), (Fig. 3.6), for a project of single output, then the average cost, \( AC \), is given by

\[
AC = \frac{TC}{Y} = \frac{f(Y)}{Y}
\]

Average cost curves are usually U-shaped. They decrease initially because of economies of scale, and increase again, as production becomes very large.
The marginal cost, $MC$, is given by

$$MC = \frac{df(Y)}{dY} = f'(Y)$$

and is represented by the slope of the total cost curve. The slope represents the change in total cost with a unit change in the output. Unless the market price exceeds the marginal cost, no firm would produce an extra unit, and therefore, the rising limb of the marginal cost curve is a supply curve. It indicates the price necessary to induce an extra unit of production.

The total cost of production of an output, $Y$, is the area of the marginal cost curve to the left of $Y$. It may be noted that the marginal cost curve intersects the average cost curve at the latter’s minimum, as

$$AC = f(Y)/Y$$

is minimum when $df(f(Y)/Y)/dY$ is zero, or

$$\left\{Yf'(Y) - f(Y)\right\}Y^2 = 0$$

or

$$f'(Y) = f(Y)/Y$$

i.e.

$$MC = AC$$

Similarly, the marginal benefit is the change in the total benefit per unit of output. The marginal benefit curve is the demand curve because no buyer will purchase an extra unit unless it provides him a value that exceeds the cost incurred. It indicates the maximum price that a firm can afford to spend to acquire an extra unit. Marginal curves, like average curves, are generally U-shaped, but justified more to the left (James and Lee, 1971).

3.4.3 Benefit and Cost Estimation

As mentioned earlier, benefits and costs need to be estimated in monetary units. Depending on the ease with which such estimates may be arrived at, there could possibly be four different situations in relation to the availability of a market, and its prices representing marginal social values.

1. Market prices exist and reflect true marginal social values, such as in the case of pure competition (ideal condition).
2. Market prices exist but, for various reasons, do not reflect marginal social values. Example: Subsidized agricultural commodities.

3. Market prices are essentially non-existent, but it is possible to simulate a market-like process to estimate what users (or consumers) would pay if a market existed. Example: Outdoor recreation.

4. No real or simulated market-like process is conceivable. Example: Historic monuments, temples, scenic amenities, etc.

For the first three categories (1, 2, 3) mentioned, benefits and costs can be measured as the aggregate net willingness to pay of those affected by the project.

**Willingness-to-Pay Criterion** The concept of willingness-to-pay is based on the common experience that a person will not buy a commodity at a cost if, in his view, the value that he accrues by using the commodity is any less than the price that he has to pay in acquiring it. In other words, the net benefit that accrues to him (over and above the cost he pays for it) is what induces him to purchase the commodity in the market. Assume (Loucks et al., 1981), all people that are benefited by the plan X are willing to pay $B(X)$ rather than forego the project. This represents the aggregate value of the project to the beneficiaries. Let $D(X)$ equal the amount that all the nonbeneficiaries of plan X are willing to pay to prevent it from being implemented. The aggregate net willingness to pay, $W(X)$, for plan X, is equal to the difference between $B(X)$ and $D(X)$, i.e. $W(X) = B(X) - D(X)$.

The rationale implied in the willingness-to-pay criterion is that if $B(X) > D(X)$, the beneficiaries could compensate the nonbeneficiaries and everyone would benefit from the project. This, however, has some implications such as the marginal social value of income to all affected parties is the same. If the beneficiaries are essentially rich and the nonbeneficiaries are poor, then $B(X)$ may be larger than $D(X)$ simply because the beneficiaries can afford to pay more than the nonbeneficiaries.

**Market Prices Equal Social Values** Let us assume that the price demanded is $p(x)$, when the quantity available in the market is $x$ (demand curve given). Let there be an increase in the level of the output from $x_1$ without the project to $x_2$ with the project. Then the willingness-to-pay (which is a proxy for the accrued benefits) is given by the area under the demand curve for the increased portion of the output, Fig. 3.7.

![Demand Curve](image_url)

*Fig. 3.7 | Increase in Output with the Project*
Willingness to pay = \int_{x_1}^{x_2} p(x)dx (3.2)

which is the area under the demand curve between \( x = x_1 \) and \( x = x_2 \). This is an estimate of the benefit attributable to the project under contemplation.

**Example 3.4.1** The demand curve for recreation at a reservoir site is determined as \( 4P + Y = 30 \), where \( Y \) is the annual demand curve and \( P \) is price in appropriate units. The annual demand was 10 without the reservoir and is expected to increase to 20 with the construction of the reservoir. Estimate the benefits of recreation arising from the construction of the reservoir.

The demand curve plots as a straight line, \( EF \), as shown in the following figure:

![Demand Curve for Recreation](image)

The area under the demand curve between \( y = 10 \) and \( y = 20 \) gives the total willingness to pay for recreation or, in other words, the benefits due to recreation at the reservoir. The area \( ACDB = \frac{5.0 + 2.5}{2} 	imes 10 = 37.50 \). The benefits are thus estimated to be 37.50.

**Market Prices do not Equal Social Values** There are several procedures that can be used depending on the situation. A rather common method is to estimate the cost of the least expensive alternative and project it as the benefit. However, the alternative cost approach is meaningless in the absence of proof that the second best alternative would be built if the best were not. As an example, let the benefits from generating hydroelectricity be estimated using this approach. Let the alternatives considered be geothermal or nuclear sources, both of which, let us assume, are more expensive than the hydroelectric scheme. Then the cost of the least expensive alternative of the two sources will be higher than the cost of the proposed hydroelectric scheme itself, and hence the proposal will have a benefit cost ratio higher than one. However, this approach to benefit estimation is valid only when there is a commitment to actually build the less expensive of the geothermal and the nuclear schemes, in case the hydroelectric scheme is not built. Therefore, this method is rather tricky in that it is possible to propose and justify a plan by choosing expensive alternatives (which cost higher than the proposed plan), and therefore, should be used with extreme caution.

The following example shows how a demand curve can be derived to estimate the value of outdoor recreation. This is a case where market prices do not exist.
but a market-like process can be simulated. There are many underlying assumptions and limitations of the approach used herein, but the example illustrates the conceptual approach that may be used in cases where an imputed demand curve is needed to estimate recreation benefits.

Example 3.4.2 A recreation area is proposed to be developed near a reservoir, which will serve the population from two towns, Town A and Town B. Town A has a population of 50,000 and Town B of 150,000. It is estimated from surveys that 150,000 visits per year will be made at a travel cost of Rs 10 per visit from town A, and 300,000 visits per year will be made at a travel cost of Rs 20 from Town B. We have to construct a demand curve for recreation and estimate the annual benefits due to recreation.

The data are tabulated as follows:

<table>
<thead>
<tr>
<th>Population</th>
<th>Visits/yr</th>
<th>Visits/capita</th>
<th>Cost/visit (Rs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Town A</td>
<td>50,000</td>
<td>150,000</td>
<td>3</td>
</tr>
<tr>
<td>Town B</td>
<td>150,000</td>
<td>300,000</td>
<td>2</td>
</tr>
</tbody>
</table>

We see that three visits per capita will be made at a cost of Rs 10 per visit, and two visits per capita will be made at Rs 20. Thus we have two points that can be plotted, as in the following figure.

We shall assume that the visitation rate of the population from both the towns depends only on the total cost incurred per visit, and estimate the user response for increased levels of total cost (travel cost + additional cost such as admission cost).

First consider an added cost of Rs 10 over and above the travel cost making it Rs 20 per visit from Town A, and Rs 30 per visit from Town B.

It is assumed that the relationship between the cost/visit and expected number of visits per capita will be the same as indicated in the figure.

Now we shall estimate user response based on the total cost per visit (travel cost + additional cost). At an additional cost of Rs 10 per visit the total...
cost per visit will be Rs 20 from town A and Rs 30 from Town B. It is seen that only two visits/capita will be made at Rs 20, and only one visit/capita at Rs 30, from the previous figure. It is also seen that at a total cost of Rs 40 per visit, no visits will be made from either town. Also, where there is no addition to travel cost, the total number of visits per year will be 150,000 from Town A + 300,000 from Town B making it a total of 450,000 visits per year.

The following table shows the number of visits per year from each town and the total number of visits per year (recreation demand) that can be expected to be made at different levels of added cost.

<table>
<thead>
<tr>
<th>Total cost</th>
<th>No. of visits/capita</th>
<th>Total no. of visits/yr</th>
</tr>
</thead>
<tbody>
<tr>
<td>Added Cost = 0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Town A</td>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>Town B</td>
<td>20</td>
<td>2</td>
</tr>
<tr>
<td>Added Cost = 10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Town A</td>
<td>20</td>
<td>2</td>
</tr>
<tr>
<td>Town B</td>
<td>30</td>
<td>1</td>
</tr>
<tr>
<td>Added Cost = 20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Town A</td>
<td>30</td>
<td>1</td>
</tr>
<tr>
<td>Town B</td>
<td>40</td>
<td>0</td>
</tr>
<tr>
<td>Added Cost = 30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Town A</td>
<td>40</td>
<td>0</td>
</tr>
<tr>
<td>Town B</td>
<td>50</td>
<td>0</td>
</tr>
</tbody>
</table>

The following figure shows the additional cost vs. total number of visits/yr due to recreation, which is an estimate of the demand for recreation at the reservoir, or the demand curve.

The total benefits due to recreation at the reservoir is the area under the demand curve, which is Rs 5,250,000/year.

No Market Process In the absence of any market-like process, real or simulated, it is extremely difficult to quantify benefits in monetary terms. The benefits associated with aesthetics, for example, are considered intangible.
Negative benefits due to submergence of temples or other places of worship and environmental impacts of a long-lasting nature belong to this category.

**Project Size** A practical goal in planning a water resource project is to plan the size of the project, such that it yields the maximum net benefits, Fig. 3.8. At the optimum output level, the slopes of the benefit and cost curves should be the same, i.e. marginal benefit should equal marginal cost.

Water is supplied from a project for two types of users: rural and urban. The benefits to rural community are given by 
\[ B_r = 30y_r - \frac{3}{2}y_r^2 \]
and those to the urban community are given by 
\[ B_u = 10y_u - \frac{2}{8}y_u^2 \]. If the total cost of the project is 
\[ C = \frac{Y^2}{2} + 2Y \], where \( Y \) is the aggregate demand, determine the optimum level of total water supply. Also determine the corresponding components of rural and urban water supply levels.

Rural
\[ B_r = 30y_r - \frac{3}{2}y_r^2 \]
The demand curve is given by the marginal benefit curve.
\[ P = \frac{dB_r}{dy} = 30 - 3y_r \text{ or } P + 3y_r = 30 \]

Urban
\[ B_u = 10y_u - \frac{1}{8}y_u^2 \]
The demand curve for urban use is given by
\[ P = \frac{dB_u}{dy} = 10 - y_u/4 \text{ or } 4P + y_u = 40 \]

For these two types of demand, the aggregate demand curve is given by the line \( AEF \) in the figure in Example 3.2.1. This line is the marginal benefit curve.

**Cost Curve**
- The Total Cost \[ C = \frac{Y^2}{2} + 2Y \]
- Marginal Cost, \[ MC = \frac{dC}{dy} = Y + 2 \]
At optimum level of water supply $Y$, the marginal benefit and marginal cost should be equal. In other words, determine the intersection of the curve \( P = Y + 2 \) and the aggregate demand curve (the line $AEF$ in the figure in Example 3.2.1).

Assuming $Y \geq 20/3$, the combined demand curve is given by $P = \frac{150 - 3Y}{13}$.

Equating this to the marginal cost gives

\[
P = \frac{150 - 3Y}{13} = Y + 2; \quad Y = \frac{31}{4} (\geq 20/3).
\]

Optimal Price $P$ (corresponding to $Y = \frac{31}{4}$) = $\frac{39}{4}$.

Since the marginal price $P$ should be the same for both types of users, the individual demands can be obtained from the individual demand curves at $P = \frac{39}{4}$.

Rural: $P = \frac{39}{4} = (30 - 3y_r)$ or $y_r = \frac{81}{12}$.

Urban: $P = \frac{39}{4} = (40 - y_u)$ or $y_u = 1$

Check: Optimum Total Demand = $y_r + y_u = (\frac{81}{12}) + 1 = \frac{93}{12} = \frac{31}{4}$, as before.

REFERENCES


Further Reading

Water resource planning is a complex and interdisciplinary problem involving many stakeholders—engineers, system analysts, economists, social scientists, environmentalists, and decision-makers, to name a few. Each of these groups has its own perceptions and ideas on what should be the objectives in water resources planning. Also, within the same group, people may differ in their ideas of what exactly these should be. Maximizing aggregated net monetary benefits to all the parties that are affected by the water resources project (positively or negatively) is an important objective perceived by many, but there are other objectives relating to environmental quality, regional development, resource utilization, social well-being, etc. that are also important. Therefore, it is possible that we may have to consider multiple objectives in water resources planning. Some of these objectives can be quantified, some are very difficult to quantify, while some others cannot be quantified at all.

We shall restrict ourselves, in this book, to only those multiple objectives that can be quantified, and avoid discussion on what should be the real objectives in water resources planning, as this is simply a matter of policy. Some of the multiple objectives may conflict with one another. For example, in a reservoir project intended to satisfy both irrigation and power generation from a powerhouse located downstream of the dam, water withdrawn for irrigation is not available for power, and releases for power are not available for irrigation. Irrigation and power production, in this case, are conflicting with each other. If the reservoir is meant to serve recreation also, then, among the three objectives, irrigation, hydropower generation and recreation, each objective is in conflict with the other two. This is because all of them share the same resource, and storage used for any one purpose is not available for the other two (recreation needs storage).
We shall introduce some basic terminology and mathematical basis for multiobjective planning before discussing the popular techniques used therein. The concept of noninferior solutions is basic to the mathematical framework for multiobjective planning. A noninferior solution is one in which no increase in any objective is possible without a simultaneous decrease in at least one of the other objectives.

Consider a problem in which two objectives, \( z_1 \) and \( z_2 \), are to be maximized and let both be functions of the decision variable \( x \) (Fig. 4.1).

It is clearly seen that solutions in the range \( x < x_1 \) and \( x > x_2 \) can be eliminated from consideration, as the objective values in this range are worse than some in the range \( x_1 \leq x \leq x_2 \). This range is therefore called a noninferior range and solutions in this range are termed noninferior solutions or pareto-admissible solutions. Each of these solutions is such that one objective value cannot be increased unless the other objective value is simultaneously decreased. In some cases, the objectives cannot be measured in monetary terms and they may just be expressed in units of their physical output.

Each of the solutions in the noninferior range gives the maximum value of any one objective for a given value of the other. Multiobjective problems, in general, do not have an ‘optimal solution’ per se. Therefore, the noninferior solutions are important. Each of these solutions may be interpreted as an outcome of putting the resource to its maximum use, or operating the system at its maximum efficiency. The decision-maker should look at these ‘efficient’ solutions and pick one, which in some sense is the ‘best’ (by a definition). This is called the best solution, best compromise solution, or the preferred solution.

The product transformation curve, or the efficiency frontier referred to in Section 3.3.1 is the boundary of the objective space containing the noninferior set of solutions. In a general problem, there can be many objectives and many plans (each containing a set of decisions). While the same objective may have different values for different plans, the same plan may result in different values for different objectives.
Let $X$ be a vector of decision variables, $X = (x_1, x_2, x_3, \ldots, x_n)$, and $Z_j(X)$, $j = 1, 2, \ldots, p$ denote $p$ objectives, each of which is to be maximized. The multiobjective problem may be written as

\[
\text{Maximize} \quad \{Z_1(X), Z_2(X), \ldots, Z_p(X)\}
\]

subject to \quad $g_i(X) = b_i \quad i = 1, 2, \ldots, m$

The objective function in the problem is a vector consisting of $p$ separate objectives. The constraints impose technical feasibility. This objective function in the vector form can be maximized only if it can be reduced to a scalar function, using a value judgment analysis of the different components in it.

4.2 PLAN FORMULATION

There are two essential steps in multiobjective analysis: plan formulation and plan selection. Plan formulation is aimed at generating the noninferior set of solutions (or set of technologically efficient solutions), and plan selection is the process of selecting the best compromise solution. Two of the most common approaches in formulating a multiobjective planning problem are the weighting method and the constraint method.

4.2.1 Weighting Method

In the weighting method, the objective function (in the vector form) is converted to a scalar by expressing it as a weighted sum of the various objectives by associating a relative weight to each objective function. If $w_j$ is the relative weight assigned to the objective $Z_j$, then the multiobjective model is written as

\[
\text{Maximize} \quad Z = w_1Z_1 + w_2Z_2 + \ldots + w_pZ_p
\]

subject to \quad $g_i(X) = b_i \quad i = 1, 2, \ldots, m$

The relative weights, $w_j$, reflect the trade-off or the marginal rate of transformation of pairs of objective functions. These weights are varied systematically, and solutions obtained for each set of values. The solution obtained for a given set of weights gives one generated set of noninferior or efficient solutions or plans. By varying the weights in each case, a wide range of plans are obtained for further analysis before the best one is selected.

Weights imply value judgments. The determination of the set of relative weights is a complex exercise and has to be done keeping the preferences of the decision-makers in mind who, in turn, are supposed to represent the interest and preferences of the beneficiaries. This requires a study of the economic, societal and developmental priorities. For a given set of weights, however, it is easy to infer the relative values of the various objectives considered in the analysis.

One major limitation of the weighting approach is that it cannot generate the complete set of efficient plans unless the efficiency frontier is strictly convex. If a part of it is concave, then, only the end points (and none on the curve in between them) of this part can be obtained by the weighting technique.
4.2.2 Constraint Method

In this method, one objective is maximized, with lower bounds on all the others.

Maximize $Z_j(X)\]
subject to $g_i(X) = b_i$ $i = 1, 2, \ldots, m$
and $Z_j(X) \geq L_k$ for all $k$ not equal to $j$

Any set of feasible values of $L_k$ resulting in a solution with binding constraints gives an efficient alternative (solution). If the constrained method of formulation can be solved using linear programming, it is particularly useful to conduct sensitivity analysis to infer the implied tradeoffs for given right-hand side values of the binding constraints. The dual variables of the binding constraints with $L_k$ on the right-hand side are the marginal rates of transformation of the objectives $Z_j(X)$ and $Z_k(X)$.

Example 4.2.1 Basic Problem Statement, following Loucks et al. 1981

Let objective $Z_1(X) = 5X_1 - 4X_2$ and $Z_2(X) = -2X_1 + 8X_2$. Both are to be maximized. Assume the constraints on the variables $X_1$ and $X_2$ are

\begin{align*}
-X_1 + X_2 & \leq 6 \\
X_1 & \leq 12 \\
X_1 + X_2 & \leq 16 \\
X_1 & \leq 8 \\
X_1, X_2 & \geq 0
\end{align*}

1. Plot a pareto chart for admissible or noninferior solutions in decision space.
2. Plot the efficient combinations of $Z_1$ and $Z_2$ in objective space.
3. Maximize $Z_1(X)$ and $Z_2(X)$ using the weighting method, given the weights associated with $Z_1$ and $Z_2$ are 1 and 2 respectively, and illustrate the method in decision and objective space.
4. Illustrate the constraint method of defining all efficient solutions in the decision space.

Solution:

1. The decision space is plotted in the following figure. By plotting the constraint boundary lines, we find that the boundary of the feasible region is $OBCDEF$. The lines of constant $Z_1$ and $Z_2$ will be parallel to the $Z$ lines shown in the figure. However, the line of maximum $Z_1$ passes through the point $F(12,0)$, and that of maximum $Z_2$ through $C (2,8)$. This may be verified by comparing the slopes of the objective lines with those of the constraint lines. For example, the slope of the objective function line, $Z_1$, is $5/4 (dx/dx_1 = 5/4)$, and the point on the boundary of the feasible region, farthest from the origin, which a line with this slope meets, is $F$. Similarly, $Z_2$ is a maximum at the point $C$.

Therefore, the noninferior set of solutions are represented by the line $CDEF$. Note that the segment $BC$ does not contain noninferior solutions, as both objectives $Z_1$ and $Z_2$ can be increased along $BC$ ($Z_1$ can be increased from $-24$ to $-22$, and $Z_2$ from 48 to 60, by moving from $B$ to $C$).
2. Evaluating the values of $Z_1(X)$ and $Z_2(X)$ at $C$, $D$, $E$, and $F$, the line of efficient combinations of $Z_1$ and $Z_2$ in the objective space is plotted. This is the line $CDEF$ in the following figure.

3. Weighting method

Maximize $Z = w_1Z_1 + w_2Z_2$
with $w_1 = 1$ and $w_2 = 2$

$Z = Z_1(X) + 2Z_2(X) = (5X_1 - 4X_2) + 2 (-2X_1 + 8X_2)$

$= X_1 + 12X_2$

Objective is to maximize $Z = X_1 + 12X_2$, subject to the given constraints.

Decision space: The $Z$ line, $Z = X_1 + 12X_2$, has a slope of $-1/12$, in the decision space and $Z$ is found to have a maximum value equal to 104 on the boundary of the decision space at $D(8,8)$. [The student may verify this.]
Objective space: The objective function line \( Z = Z_1 + 2Z_2 \) has a slope of \(-1/2\). The value of \( Z \) will be maximum at \( D(8, 48) \) in the objective space, and is equal to \( Z = 8 + 2(48) = 104 \).

4. **Constraint method**

The problem may be solved using LP.

Maximize \( Z_1(X) \)

subject to \( Z_1(X) \geq L_2 \), along with the other constraints.

Any optimal solution for an assumed value of \( L_2 \) is a noninferior solution, if the constraint with \( L_2 \) on the right-hand side is binding. By varying the value of \( L_2 \), we get different noninferior solutions.

An easier way to identify different noninferior solutions is to solve the following problem using LP.

Maximize \( Z_1 = 5X_1 - 4X_2 \)

subject to \(-X_1 + 4X_2 = L_2 \)
\(-X_1 + X_2 \leq 6 \)
\(-X_1 \leq 12 \)
\(X_1 + X_2 \leq 16 \)
\(X_1 \leq 8 \)
\(X_1, X_2 \geq 0 \)

Solve for different values of \( L_2 \) for which the problem is feasible. Each set of the maximized \( Z_1 \) value and the corresponding \( L_2 (Z_2 \) value) defines one noninferior solution. The entire set of noninferior solutions may be obtained by solving the problem for all feasible values of \( L_2 \) (problems of this type can be easily solved on a PC using PC software for LP such as LINGO).

**Problems**

4.2.1 A reservoir is planned both for gravity and lift irrigation through withdrawals from its storage. The total storage available for both uses is limited to 5 units each year. It is decided to limit the gravity irrigation withdrawal in a year to 4 units. If \( X_1 \) is the allocation of water to gravity irrigation and \( X_2 \) the allocation for lift irrigation, two objectives are planned to be maximized and are expressed as

\[ Z_1 = 3X_1 - 2X_2 \]
\[ Z_2 = -X_1 + 4X_2 \]

1. Formulate a multiobjective planning model using weighting approach with weights for gravity and lift irrigation withdrawals being \( w_1 \) and \( w_2 \) respectively. Plot the decision space and the objective space and determine the optimal share of withdrawals for gravity and lift irrigations, if \( w_1 = 1 \) and \( w_2 = 2 \). (i) \( w_1 = 1 \) and \( w_2 = 2 \).
   (ii) \( w_1 = 2 \) and \( w_2 = 1 \).

2. Formulate the problem using the Constraint method.

   \[ \text{Ans: (i) } X_1 = 0; X_2 = 5. \text{ (ii) } X_1 = 4; X_2 = 0 \text{ to } 1 \]
4.2.2 Refer to Example 4.2.1 illustrated earlier. Assume that while maximizing the weighted objective function, \( Z = w_1 Z_1 + w_2 Z_2 \), the decision-maker chose the point \( E (X_1 = 12, X_2 = 4) \) in the decision space as the preferred solution. Determine the value or range of values of the marginal rate of substitution of the objective \( Z_2 \) to \( Z_1 \). Interpret what it means in the perception of the decision maker.

\[
\text{Ans: } -10/9 > \frac{w_1}{w_2} > -2
\]

4.3 PLAN SELECTION

Plan selection depends essentially on the relative preferences of the decision maker to the various objectives to be maximized. In the weighting method, e.g., if the relative weights attached to each objective are specified then the particular solution corresponding to those specified weights will be the selected plan. However, determination of these relative weights itself is a problem in multiobjective planning. Necessarily, multiobjective planning is an interactive and iterative process involving decision makers and systems analysts. Quantification of intrinsic preferences to different objectives is necessary for multi-objective planning and this is where the systems analyst’s role comes into play. There are a number of plan generation and selection techniques that warrant a more elaborate discussion than is possible in an elementary book like this. The publications mentioned at the end of the chapter may be consulted for further reading on this.

REFERENCE


Further Reading

PART 2

Model Development

- Reservoir Systems—Deterministic Inflow
- Reservoir Systems—Random Inflow
A reservoir is a storage structure that stores water in periods of excess flow (over demand) in order to enable a regulation of the storage to best meet the specified demands. Modelling a reservoir system, in general, is specific to the system, and the assumptions one may use to make the model formulation simple enough for problem-solving through known techniques in systems analysis. This chapter illustrates how a reservoir may be modelled using deterministic inputs. Model formulations for two important aspects of reservoir modelling are discussed: reservoir sizing and reservoir operation.

**5.1 RESERVOIR SIZING**

The annual demand for water at a particular site may be less than the total inflow there, but the time distribution of the demand may not match the time distribution of inflow, resulting in surplus in some periods and deficit in some other periods of the year. A reservoir serves the purpose of temporarily storing water in periods of excess inflow and releasing it in periods of low flow so that the demands may be met in all periods. The problem of reservoir sizing involves determination of the required storage capacity of the reservoir when inflows and demands in a sequence of periods are given. The total storage can be divided into three components: dead storage (for accumulation of sediments), active storage (for conservation purpose such as water supply and hydropower production), and flood storage (for reducing flood peak). While each of these components may be determined by separate modelling studies, we confine ourselves in this section only to the determination of the active storage capacity of the reservoir.

The inflow to the reservoir is in fact a random variable. The problem gets complicated if the randomness of the inflow has to be taken into account. Let us consider the simpler case of deterministic inflow for the purpose of the present discussion, and also assume that the given inflow sequence repeats
itself. If the length of the inflow sequence is a year, it means that the inflow in a given (within-the-year) period is the same in all the years.

One common method, extensively used in practice, is to determine the active storage capacity using the Rippl diagram or the mass diagram by plotting cumulative inflow with time. The method involves finding the maximum positive cumulative difference between a sequence of reservoir releases (equal to demands) and historical inflows over a sequence of time periods in which the demand is constant. For details of this method, the reader may refer any standard textbook on hydrology. The procedure is shown in Fig. 5.1.

If the demand is constant in each time period, the method is quite simple to apply. When the demand varies across time periods, the procedure requires a plot of the cumulative deficits in time from the period in which a deficit sets in, for the duration of the deficit, and finding the maximum deficit among all such durations. The total deficit duration containing this maximum deficit is known as the critical period.

This method is still cumbersome compared to a simple analytical technique, known as sequent peak method, which is described next. If we ignore the evaporation losses in the reservoir, the sequent peak method is quite simple. We shall first assume this situation and look at the algorithm to determine the reservoir capacity by the sequent peak method (Section 5.1.1). Later, we shall see how the reservoir losses can be incorporated into the method to determine the active storage capacity of the reservoir.

5.1.1 Sequent Peak Analysis Neglecting Evaporation

The sequent peak analysis can be applied for constant or varying demands in time. In this method, we find the maximum cumulative deficit over adjacent sequences of deficit periods, and determine the maximum of these cumulative deficits. The inflow sequence is assumed to repeat and the analysis is carried
out over two cycles, or two consecutive inflow sequences. If the critical period lies towards the end of an inflow sequence, carrying out the analysis over two cycles ensures the capture of the maximum value of the cumulative deficit, which really is the required active storage capacity. The sequent peak algorithm is as follows:

Let $t$ denote the time period, $Q_t$ the inflow, and $R_t$ the required release or demand in period $t$. Let $K_t$ be defined as follows:

$$K_t = \begin{cases} K_{t-1} + R_t - Q_t & \text{if positive,} \\ 0 & \text{otherwise,} \end{cases}$$

with $K_0$ set equal to zero ($K_0 = 0$).

Or, $K_t$ may be expressed conveniently as the maximum of zero and $K_{t-1} + R_t - Q_t$, as

$$K_t = \max \{0, K_{t-1} + R_t - Q_t\}$$

The values of $K_t$ are computed for each period $t$ for two cycles or successive inflow sequences.

Let $K^* = \max \{K_t\}$ over all $t$. Then $K^*$ is the required active storage capacity of the reservoir.

It may be noted that if the value of $K_t$ is zero at the end of the last period of the first cycle, then computations over the second cycle are not necessary. This happens when the critical period is entirely contained in the first cycle itself. Otherwise, the second cycle computations are required. Here again, computations of $K$ for all the periods in the second cycle also are not necessary, beyond a time period $t$ for which the value of $K$ is exactly the same as that in the first cycle for the corresponding period (as in Table 5.1 in Example 5.1.1).

It may be noted that it is immaterial whether or not the historical inflow sequence coincides with a calendar year or a water year. The important feature is that computation of $K_t$ over two consecutive cycles (inflow sequences) will ensure that the correct value of $K^*$ is determined.

The sequent peak method is just an analytical procedure to determine the maximum cumulative deficit that occurs over time, given the inflows and demands across time periods. The method is very similar to the mass diagram approach, when the evaporation losses in the reservoir are neglected.

It is obvious that if the total annual demand exceeds the total annual inflow, no amount of reservoir storage would satisfy the full demand in all the periods of a year. This is because of inflow limitation, or lack of available water.

### Example 5.1.1
(Reservoir capacity with evaporation loss neglected)

Determine the required capacity of a reservoir whose inflows and demands over a 6-period sequence are as given below (release, $R_t = \text{demand}, D_t$).

<table>
<thead>
<tr>
<th>Period, $t$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inflow, $Q_t$</td>
<td>4</td>
<td>8</td>
<td>7</td>
<td>3</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Demand, $D_t = R_t$</td>
<td>5</td>
<td>0</td>
<td>5</td>
<td>6</td>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

It is assumed that the sequence repeats itself. In this case, total inflow = total demand in the six periods of the sequence and is equal to 24. Since
evaporation loss is neglected, it is possible to determine the required storage to meet the demands in full. In this case, release equals demand in each period and there is no spill. Table 5.1 illustrates the computations using the sequent peak algorithm.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$R_t$</th>
<th>$Q_t$</th>
<th>$K^*$</th>
<th>$K^* = K^* + R_t - Q_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>8</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>7</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>3</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>0</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>4</td>
<td>9</td>
<td>$10^9$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>8</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>7</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3 repeats</td>
</tr>
</tbody>
</table>

* $K$ is max $\{0, K_{t-1} + R_t - Q_t\}$
* $\max \{K_t\} = K^*$, the required capacity

The computations in the second cycle repeat after period 5 (hence, are not shown). The required capacity of the reservoir is $\max \{K_t\} = 10$. Note: A reservoir of capacity of $K^* = 10$ will be full at the end of the third period, and will be empty at the end of the first period of the following sequence (analyse why?). The intervening period is the ‘critical period.’

**Constant demand:** If the demand in each period is constant (say, for instance, the water supply demand for a town), then $R_t = R$ for all $t$ in the above formulation. It is to be noted that in the absence of losses, the maximum constant demand, $R$, which can be met from a given sequence of inflows is equal to the average of the inflows, $Q_t$, over all $t$. A constant demand larger than this magnitude cannot be met, whatever be the reservoir capacity, due to inadequate quantum of inflow. In the example illustrated above, the maximum constant demand that can be met in a time period is $24/6 = 4$ units of water. Any higher value is infeasible due to limitation of the total quantum of inflow, no matter what the reservoir capacity is.

### 5.1.2 Sequent Peak Analysis Considering Evaporation Loss

The sequent peak method can be modified (Lele, 1987) to take into account storage-dependent losses in the determination of the reservoir capacity, with some additional data and computations. The additional data required are the evaporation rate in each time period and the area capacity curve of the reservoir. A procedure using sequent peak method is presented in this section, which takes into account evaporation losses in the reservoir.
The reservoir storage continuity may be written as

\[ S_t + Q_t - R_t - E_t = S_{t+1} \]

where \( S_t \) is the active storage at the beginning of period \( t \), \( Q_t \) is the inflow during period \( t \), \( R_t \) is the release during period \( t \), and \( E_t \) is the evaporation loss during period \( t \).

Evaporation loss (in volume units) in a period is given by the product of the evaporation rate (in depth units) and the average water spread area (average of the water spread areas at the beginning and at the end of the period) of the reservoir in that period. The water spread area is a function of the total storage (dead + active) in the reservoir. The reservoir level will always be above the dead storage level. If water spread area is plotted against reservoir storage, the relation between the water spread area above the dead storage level and the active storage (storage above the dead storage) can, in most cases, be approximated by a straight line (Loucks et al., 1981), as in Fig. 5.2.

![Fig. 5.2 Approximation of Reservoir Area vs. Storage Relationship](image)

The evaporation loss in period \( t \) then is given by

\[ E_t = e_t A_o \]

where \( A_t = A_o + a (S_t + S_{t+1})/2 \), or

\[ E_t = L_t + a_t (S_t + S_{t+1}) \]

where \( L_t \) is the fixed evaporation loss = \( e_t A_o \), \( e_t \) is the evaporation rate in period \( t \), \( A_o \) is water surface area at the top of the dead storage level, \( a_t = ae_t/2 \), and ‘\( a \)’ is the surface area per unit active storage, or the slope of the straight line segment with water surface area (y-axis) plotted against reservoir storage (x-axis), Fig. 5.2.

The storage continuity equation, taking into account evaporation loss, thus becomes

\[ (1 - a_t) S_t + Q_t - L_t - R_t = (1 + a_t) S_{t+1} \]

This equation defines the relationship between the active storages at the beginning and at the end of period \( t \). The algorithm (Lele, 1987) to determine the required capacity of the reservoir is given by the following steps.

**Step 1 Initialization**

1. Record the data: \( Q_t, R_t, e_t, A_o \), and \( a \).
2. \( E_t = 0 \) for all \( t = 1, 2, \ldots, T \), where \( T \) is the last period of the inflow sequence.
Step 2  Sequent peak procedure
1. \( K_0 = 0 \)
2. Calculate \( K_t = \max \{0, K_{t-1} + R_t + L_t - Q_t\} \) for all \( t = 1, 2, \ldots, T \)
3. If \( K_T = K_0 \), then go to 4; else if this is the first iteration in Step 2, then put \( K_0 = K_T \) and go to 2; else STOP. Sequent peak analysis failed as the gross demand is greater than the average inflow.
4. Find the period for which \( K_t \) is maximum over all \( t = 1, 2, \ldots, T \), and call this period \( ICRIT \). This is the period at the end of which the reservoir will be empty.
5. Search backward from \( t = ICRIT \) until \( K_t = 0 \) is encountered for the first time. Call this period, for which \( K_t = 0 \), as \( IOVF \). This is the period at the end of which the reservoir will be full.

Step 3  Recalculation
1. \( S_{ICRIT+1} = 0 \).
2. From \( t = ICRIT \), move backward to \( t = IOVF + 1 \), calculate \( S_t = [S_{t+1}(1 + a_t) + R_t + L_t - Q_t]/(1 - a_t) \).
3. \( K^* = S_{IOVF+1} \).

Step 4  Checking
1. For \( t = ICRIT + 1 \) forward to \( t = IOVF - 1 \), calculate
   \( S_{t+1} = \min \{K^*, [S_t(1 - a_t) + Q_t - R_t - L_t]/(1 + a_t)\} \)
2. If any \( S_t \) is negative, then go to 3 (the sequent peak procedure should be repeated with new values of evaporation losses, \( E_t \), using the current values of \( S_t \); else STOP. \( K^* \) is the required active storage.
3. Calculate \( E_t = e_t [a(S_t + S_{t+1})/2 + A_0] \) for all \( t = 1, 2, \ldots, T \) and go back to Step 2.

The required reservoir capacity will be naturally higher, when the evaporation loss is taken into account, than when it is not.

Example 5.1-2  (Reservoir capacity with evaporation loss considered)
The monthly inflows (\( Q_t \)) and demands (\( D_t \)), in Mm\(^3\), and evaporation rate, \( e_t \), in mm for a reservoir are given below.

<table>
<thead>
<tr>
<th></th>
<th>June</th>
<th>July</th>
<th>Aug</th>
<th>Sept</th>
<th>Oct</th>
<th>Nov</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q_t )</td>
<td>70.61</td>
<td>412.75</td>
<td>348.40</td>
<td>142.29</td>
<td>103.78</td>
<td>45.00</td>
</tr>
<tr>
<td>( D_t )</td>
<td>51.68</td>
<td>127.85</td>
<td>127.85</td>
<td>65.27</td>
<td>27.18</td>
<td>203.99</td>
</tr>
<tr>
<td>( e_t )</td>
<td>231.81</td>
<td>147.57</td>
<td>147.57</td>
<td>152.14</td>
<td>122.96</td>
<td>121.76</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Dec</th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
<th>Apr</th>
<th>May</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q_t )</td>
<td>19.06</td>
<td>14.27</td>
<td>10.77</td>
<td>8.69</td>
<td>9.48</td>
</tr>
<tr>
<td>( D_t )</td>
<td>203.99</td>
<td>179.47</td>
<td>89.76</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( e_t )</td>
<td>99.89</td>
<td>97.44</td>
<td>106.14</td>
<td>146.29</td>
<td>220.97</td>
</tr>
</tbody>
</table>

Reservoir data: Area at dead storage level, \( A_0 = 37.01 \) Mm\(^2\); Slope, \( a = 0.117115 \).
The computations are shown in the following table (Table 5.2).
The required reservoir capacity, when evaporation losses are neglected, is 588.11 Mm$^3$; whereas, with evaporation losses taken into account, the required capacity works out to 617.986 Mm$^3$.

Example 5.3.1

1. Compute the active storage capacity of a reservoir to supply a constant maximum annual yield, given the following sequence of annual flows: {8, 4, 6, 2, 4, 6}
2. Assume a year has two distinct seasons, wet and dry. Eighty per cent of annual flow occurs in the wet season each year, and eighty per cent of annual yield is demanded in the dry season each year. Determine the percentage increase in the required storage capacity compared to 1 above.

Solution:
Assuming the annual inflow sequence repeats, the maximum constant yield possible (without losses) = average annual inflow = $(8 + 4 + 6 + 2 + 4 + 6)/6 = 5$.

1. The following table shows the computations of $K_t$ in each period.

<table>
<thead>
<tr>
<th>Period, t</th>
<th>$R_t$ = release</th>
<th>$Q_t$ = inflow</th>
<th>$K_t$ = max {0, $K_{t-1}$ + $R_t$ – $Q_t$}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>4</td>
<td>4 (maximum)</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>8</td>
<td>0</td>
</tr>
</tbody>
</table>

In this example, computations can stop after period 1 in the second cycle as the value of $K_t$ for period 1 is the same (= 0) as that for the period 1 in the earlier cycle. The entries for the subsequent periods simply repeat (as those for the corresponding periods in the first cycle).

Required active storage capacity = max {$K_t$} = 4.

With this capacity, the reservoir will be full at the end of the first and third periods and will be empty at the end of the fifth period (verify). **Hint:** Examine $S_t + Q_t - R_t = S_{t+1}$ over a few cycles (sequences) starting from any initial storage, $S_t$ in any period, $t$, till steady state is reached, i.e., $S_t$ for given $t$ is the same in different cycles.

2. Inflow in the wet season = 0.8 (annual flow) each year, and inflow in the dry season = 0.2 (annual flow) each year.

Demand in the wet season = 0.2 (5) = 1 each year, and demand in the dry season = 0.8 (5) = 4 each year.
Sequent peak method is applied to the 6-year (12 seasons) sequence. For the sequent peak method, one cycle, therefore, has 12 seasons (6 years).

The following table shows the computations.

\[ K_{t+6} = K_0 = 0 \]

<table>
<thead>
<tr>
<th>Period</th>
<th>( R_t )</th>
<th>( Q_t )</th>
<th>( K_t = \max {0, K_{t-1} + R_t - Q_t} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 wet</td>
<td>1</td>
<td>6.4</td>
<td>0.0</td>
</tr>
<tr>
<td>Dry</td>
<td>4</td>
<td>1.6</td>
<td>2.4</td>
</tr>
<tr>
<td>2 wet</td>
<td>1</td>
<td>3.2</td>
<td>0.2</td>
</tr>
<tr>
<td>Dry</td>
<td>4</td>
<td>0.8</td>
<td>3.4</td>
</tr>
<tr>
<td>3 wet</td>
<td>1</td>
<td>4.8</td>
<td>0.0</td>
</tr>
<tr>
<td>Dry</td>
<td>4</td>
<td>1.2</td>
<td>2.8</td>
</tr>
<tr>
<td>4 wet</td>
<td>1</td>
<td>1.6</td>
<td>2.2</td>
</tr>
<tr>
<td>Dry</td>
<td>4</td>
<td>0.4</td>
<td>5.8</td>
</tr>
<tr>
<td>5 wet</td>
<td>1</td>
<td>3.2</td>
<td>3.6</td>
</tr>
<tr>
<td>Dry</td>
<td>4</td>
<td>0.8</td>
<td>6.8 (maximum)</td>
</tr>
<tr>
<td>6 wet</td>
<td>1</td>
<td>4.8</td>
<td>3.0</td>
</tr>
<tr>
<td>Dry</td>
<td>4</td>
<td>1.2</td>
<td>5.8</td>
</tr>
</tbody>
</table>

The required storage capacity is \( \max \{K_t\} = 6.8 \), an increase of 2.8 over the case in 1, which is an increase of \((6.8 - 4.0)/4 = 0.7\) or 70%.

This example clearly shows that, the capacity requirement increases when the variations in inflow and demands within the year periods are considered, rather than when the capacity is computed based on the annual values alone.

The sequent peak method does not require any optimization software. Evaporation accounting can easily be incorporated with some additional computations. But even without evaporation, the algorithm is not readily adaptable to a system with more than one reservoir. Mathematical programming tools provide such a capability. For example, it is more elegant to solve the problem of determining the reservoir capacity with evaporation loss taken into account using linear programming. The result, i.e. the required active storage capacity, of course, should be the same irrespective of the method used—sequent peak or linear programming. It may be noted that the linearity assumption of area with storage is not necessary in the sequent peak method as long as the evaporation loss is expressed as a known function of the storage. In LP, however, the linearity assumption simplifies incorporating the evaporation loss function easily into storage continuity relationships.
5.1.3 Reservoir Capacity Using Linear Programming

We shall discuss here how linear programming can be used to determine the required active storage capacity of a single reservoir with evaporation losses taken into account. The demand may be constant or varying with time. This is an alternate method and is more elegant than the sequent peak method for solving the same problem.

**Linear Programming Method**

It is assumed, for the purpose of present discussion, that inflows are deterministic (known) and the sequence of inflows repeats itself in time.

There are two sets of constraints to be satisfied; one relates to storage continuity and the other to the capacity. Let \( R_t \) be release (variable) and \( D_t \), the specified demand.

**Constraints**

The storage continuity constraint is

\[
(1 - a_t) S_t + Q_t - L_t - R_t = (1 + a_t) S_{t+1}
\]

for all \( t \). Also \( R_t \geq D_t \) for all \( t \).

The reservoir capacity \( K \) is a variable, which must be minimized. The objective then is to minimize \( K \).

The model formulation thus reduces to:

Minimize \( K \)

subject to

\[
(1 - a_t) S_t + Q_t - L_t - R_t = (1 + a_t) S_{t+1}
\]

for all \( t \); Also \( R_t \geq D_t \) for all \( t \).

The reservoir capacity \( K \) is a variable, which must be minimized. The objective then is to minimize \( K \).

The last constraint means that when we have a sequence of monthly inflows for a year, \( T = 12 \), \( S_{13} \) in the formulation is set equal to \( S_1 \). The purpose is to ensure that the storage at the end of the last period in the year is the same as the storage at the beginning of the first period, as the inflow sequence is assumed to be repetitive. In the storage continuity equation written above, spill, if any in period \( t \), is absorbed in the term \( R_t \).

The problem can be very easily solved on a personal computer by using LP software such as LINCO (Language for INteractive General Optimization, LINDO Systems Inc., Chicago, Illinois, USA).

**Example 5.1.1** (Reservoir capacity with evaporation neglected, using LP)

The following is the LP formulation of the problem (Example 5.1.1), solved earlier by the sequent peak method, neglecting evaporation.

<table>
<thead>
<tr>
<th>Period, ( t )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inflow, ( Q_t )</td>
<td>4</td>
<td>8</td>
<td>7</td>
<td>3</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Demand, ( D_t = R_t )</td>
<td>5</td>
<td>0</td>
<td>5</td>
<td>6</td>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

The problem formulation is:

**Objective Function:**

Minimize \( K \)

**Constraints:**

\( S_1 + 4 - 5 = S_2 \)
**Model Development**

\[ S_2 + 8 - 0 = S_3 \]
\[ S_3 + 7 - 5 = S_4 \]
\[ S_4 + 2 - 2 = S_5 \]
\[ S_5 + 0 - 6 = S_6 \]
\[ S_6 + 0 - 6 = S_1 \]

Solving using LP, the optimal solution is \( K = 10 \).

**Example 5.1.5** (Reservoir capacity with evaporation loss considered using LP.)

The example problem earlier mentioned under sequent peak method (Example 5.1.2) can now be solved using linear programming.

The monthly inflows \((Q_i)\) and demands \((D_i)\), in mm\(^3\), and evaporation rate, \(e_i\), in mm for the reservoir are given below.

<table>
<thead>
<tr>
<th></th>
<th>June</th>
<th>July</th>
<th>Aug</th>
<th>Sept</th>
<th>Oct</th>
<th>Nov</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Q_i)</td>
<td>70.61</td>
<td>412.75</td>
<td>348.40</td>
<td>142.29</td>
<td>103.78</td>
<td>45.00</td>
</tr>
<tr>
<td>(D_i)</td>
<td>51.68</td>
<td>127.85</td>
<td>127.85</td>
<td>65.27</td>
<td>27.18</td>
<td>203.99</td>
</tr>
<tr>
<td>(e_i)</td>
<td>231.81</td>
<td>147.57</td>
<td>147.57</td>
<td>152.14</td>
<td>122.96</td>
<td>121.76</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Dec</th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
<th>Apr</th>
<th>May</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D_i)</td>
<td>203.99</td>
<td>179.47</td>
<td>89.76</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(e_i)</td>
<td>99.89</td>
<td>97.44</td>
<td>106.14</td>
<td>146.29</td>
<td>220.97</td>
<td>246.75</td>
</tr>
</tbody>
</table>

Reservoir data: Area at dead storage level, \(A_0 = 37.01 \text{ Mm}^2\); Slope, \(a = 0.117115\)

Since equality form of constraints are used, it is important to distinguish between the demand, \(D_i\), and the release, \(R_i\), in formulating constraints. As the demands have to be met in every period, the lower bound for \(R_i\) is \(D_i\), or \(R_i \geq D_i\) for all \(t\), which must be specified as additional constraints in the model formulation. This inequality will take care of any spill necessary in a period in addition to meeting the demand in full.

The LP model, with notation defined earlier, therefore, is,

**Minimize** \(K\)

**subject to**

\[(1 - a_t) S_t + Q_t - L_t - R_t = (1 + a_t) S_{t+1} \text{ for all } t,\]

\[S_t \leq K \text{ for all } t,\]

\[R_t \geq D_t \text{ for all } t,\]

\[S_{T+1} = S_1\]

When solved using LP, the optimal solution for \(K = 617.986 \text{ Mm}^3\) (which is the same as \(K\) obtained in the sequent peak method, earlier, Table 5.2). Also, if \(e_i\) is set equal to zero for all \(t\), the LP solution yields \(K = 588.11 \text{ Mm}^3\), which is the required capacity when evaporation loss is neglected, as obtained earlier using sequent peak method (Table 5.2). Verify the results in both cases using a PC software such as LINGO.
Some Tips in Model Formulation Using LP

The following points are of interest in linear programming formulations discussed earlier. These have to do basically with the ways in which the reservoir storage continuity constraints are structured.

1. Equality Form of Constraints Consider the equality constraint

\[(1 - a_t) S_t + Q_t - L_t - R_t = (1 + a_t) S_{t+1}\] for all \(t\).

The constraint should imply that \(R_t\) includes the demand to be met and possible spill in the period \(t\). In order to ensure that demand in every period is satisfied in full, additional constraints, \(R_t \geq D_t\), may be introduced in the formulation, still retaining spill, if any, in period \(t\), to be absorbed in the term \(R_t\).

2. Inequality Constraints A simple alternative to using the equality constraints for storage continuity is to use the greater than or equal to sign (\(\geq\)) in place of the equality sign (=) in the continuity constraints, and substitute the demand \(D_t\) in place of \(R_t\), as

\[(1 - a_t) S_t + Q_t - L_t - D_t \geq (1 + a_t) S_{t+1}\] for all \(t\).

This form of specifying the constraints builds in flexibility to take care of spills, where necessary. It ensures that the demand is met in full and provides for any spill in periods of high flow. Note that spill can occur only when the capacity is reached. The formulation will ensure that the excess water over the demand and evaporation in any period will be stored in the reservoir itself, if the reservoir is not full (in which case there is no spill), or will be stored up to the reservoir capacity and the rest spilled. The inequality form is the most convenient form of specifying the storage constraints. The problem statement thus is:

Minimize \(K\)

subject to

\[(1 - a_t) S_t + Q_t - L_t - D_t \geq (1 + a_t) S_{t+1}\] for all \(t\)

and

\(S_t \leq K\) for all \(t\)

The spill in period \(t\), \(Spill_t\), in this formulation will be equal to

\(Spill_t = (1 - a_t) S_t + Q_t - L_t - a_t S_{t+1} - K\) if positive,

= 0, otherwise.

3. Other Equality form of Constraints (i) Specify an additional term for spill in each constraint and penalize spills in the objective function.

\[(1 - a_t) S_t + Q_t - L_t - D_t - Spill_t = (1 + a_t) S_{t+1}\] for all \(t\)

The objective function is modified as

Minimize \(K + M \sum Spill_t\), where \(M\) is an arbitrarily large number.

Here also, full demands will be met in each period and spills will be minimized, while minimizing the reservoir capacity. If the term, \(Spill_t\), is not introduced into the objective function, it is possible for \(Spill_t\) to be positive even when the reservoir does not reach the capacity at the end of the period, which is physically meaningless.
5.1.4 Storage Yield Function (Maximum Yield Problem)

In Section 5.1.3, we discussed the problem of determining the minimum reservoir capacity for given demands. A complementary problem is to determine the maximum yield (the quantum of water that can be taken out of storage per period, assumed constant) from a reservoir of given capacity. While the former is a problem in planning and design, the latter is one of determining what best can be expected from an existing reservoir. This is of interest in evaluating the performance of an existing reservoir in terms of its expected yield.

Let the objective be to maximize the yield, \( R \) (per period), from a reservoir of given capacity, \( K \). Then the following formulation for the problem can be used for LP.

Maximize \( R \)

subject to

\[
(1 - a_t) S_t + Q_t - L_t - R \geq (1 + a_{t+1}) S_{t+1} \quad \text{for all } t
\]

and

\( S_t \leq K \) for all \( t \)

with \( T + 1 = 1 \), where \( T \) is the last period.

\( K \) is given, and \( R \) is to be determined.

**Mixed Integer LP Formulation for Maximizing Yield**

Making use of integer variables in LP formulation is an elegant way to solve the problem using equality constraints with \( D_t \), and explicitly accounting for spills. The mixed integer formulation ensures that spill will not occur unless the reservoir is full. This is done by introducing additional constraints.

Introduce additional constraints into the formulation as follows:

\[
\text{Spill}_t \leq \beta \text{Spill}_{\text{max}}, \quad \text{where } \text{Spill}_{\text{max}} \text{ is the maximum possible spill in any period (as } \text{Spill}_{\text{max}} \text{ cannot exceed the maximum monthly inflow in the year, the latter may be used in place of } \text{Spill}_{\text{max}}). \text{ Alternatively, the constraints may be specified as}
\]

\[
\beta \leq \frac{S_{t+1}}{K}
\]

\( \beta \) is integer \( < 1 \)

These constraints satisfy that

when \( S_{t+1} \) is less than the capacity \( K \), then \( \beta = 0 \), and \( \text{Spill}_t = 0 \), and

when \( S_{t+1} \) is higher than the capacity \( K \), then \( \beta = \frac{S_{t+1}}{K} \text{ is greater than } 1 \). But \( S_{t+1} \) is constrained to be no greater than \( K \), because of the capacity constraint, with the result \( \beta \) will be forced to be equal to 1, in order to make \( \text{Spill}_t \) positive (in which case, \( S_{t+1} \) equals \( K \)). Then all the three sets of constraints mentioned above are consistent.

Thus the mixed integer LP formulation for maximizing yield is

Maximize \( R \)

subject to

\[
(1 - a_t) S_t + Q_t - L_t - R - \text{Spill}_t = (1 + a_{t+1}) S_{t+1} \quad \text{for all } t
\]

\( \text{Spill}_t \leq \beta M \), where \( M \) is an arbitrary large constant
Reservoir Systems—Deterministic Inflow

\[ \beta \leq S_{t+1}/K \]
\[ \beta \text{ is integer } \leq 1 \]
\[ S_t \leq K \text{ for all } t, \]

and

\[ S_{t+1} = S_t. \]

In all these cases, it must be noted that if the data are such that the total annual inflow in all the periods in the sequence is less than the sum of the demands and evaporation losses, LP obviously gives an infeasible solution. In this case, the demands cannot be satisfied in full in all periods, no matter how large is the reservoir capacity, simply because of inflow limitation. If the inflow in each period is higher than the demand, however, we need neither storage nor reservoir.

There is a maximum value of \( R \) associated with each \( K \), and a minimum required capacity, \( K \), with each \( R \). If one is given the other can be found. However, it must be noted that the problem of determining \( K \) for a given \( R \) cannot be solved using the mixed integer formulation. In the mixed integer formulation, the constraint

\[ \beta \leq S_{t+1}/K, \text{ or } \beta \leq K \leq S_t \]

becomes nonlinear when both \( \beta \) and \( K \) are variables, as in the capacity determination problem. This does not apply for the yield determination problem, as \( K \) is given and \( R \) is to be determined.

Storage Yield Function A plot of \( K \) for different values of \( R \) is called a storage yield function. This will be an increasing function of \( R \), up to some maximum feasible value of \( R \). Beyond this, the problem becomes infeasible, meaning there is not enough water to yield \( R \) in full in each period. The upper bound for \( R \) is because of the inflow limitation. Building a reservoir beyond the capacity, corresponding to the upper bound of \( R \), is a waste, as it does not contribute to any additional yield.

Model formulation for determining required reservoir capacity to meet given demands at specified reliability is presented in Section 6.3, and an example application with model formulation for multiple crops is presented in Section 7.3

### Problems

5.1.1 Determine the required capacity (\( K \)) of the reservoir for each of the given annual yields, given inflow data for a 6-year repetitive sequence, as shown in the following table. Use LP, neglecting evaporation losses. Plot the storage yield function. Verify that the required values of \( K \) are as given in the last row of the table.

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inflow</td>
<td>16</td>
<td>8</td>
<td>12</td>
<td>4</td>
<td>8</td>
<td>12</td>
</tr>
<tr>
<td>Yield</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>Min. ( K )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>8</td>
</tr>
</tbody>
</table>
5.1.2 Consider the 12-month data given in Example 5.1.5. Use LP to determine the reservoir capacity using
(a) storage continuity constraints of the greater than or equal to type, with demands as lower bounds for releases,
(b) storage continuity constraints in the equality form, incorporating demands and spills explicitly, and penalizing spills in the objective function,
(c) storage continuity constraints in the equality form, incorporating demands and spills explicitly, and using mixed integer LP,
(In all the three cases, you should get the same answer)

5.1.3 A reservoir is planned to supply water at a constant rate per season to an industry. The reservoir inflows in the four seasons of a year are 1500, 800, 1200, and 500, respectively, and repeat every year. Consider evaporation loss from the reservoir in any season to be 5% of the average storage in that season. Neglect all other losses. Formulate and solve an LP model to determine the constant maximum rate of water supply per season, if the reservoir capacity is 450 units.

\[ \text{(Ans. 938.75)} \]

5.1.4 A reservoir is to be constructed to supply water at a maximum constant rate per season for a city. The inflows in the six seasons of the year are 3, 12, 7, 3, 2, and 3, respectively. Determine the minimum required reservoir capacity using sequent peak method. Neglect all losses.

\[ \text{(Ans. 9)} \]

5.1.5 In general, if the storage continuity constraints are used as expressed below, in the LP formulation to determine reservoir capacity,
\[
(1 - a_t) S_t + Q_t - L_t - D_t \leq (1 + a_t) S_{t+1} \quad \text{for all } t,
\]
analyze the nature of the solution for feasibility.

\[ \text{(Hint: First analyze the case without evaporation losses).} \]

5.2 RESERVOIR OPERATION

A reservoir operating policy is a sequence of release decisions in operational periods (such as months), specified as a function of the state of the system. The state of the system in a period is generally defined by the reservoir storage at the beginning of a period and the inflow to the reservoir during the period. Once the operating policy is known, the reservoir operation can be simulated in time with a given inflow sequence. A number of optimization algorithms have been developed for deriving reservoir operating policies. However, the most common policy implemented in practice is the so-called standard operating policy, which is discussed first in this section. This policy by itself is not based on or derived from any optimization algorithm.

5.2.1 Standard Operating Policy

The standard operating policy (SOP) aims to best meet the demand in each period based on the water availability in that period. It thus uses no foresight on what is likely to be the scenario during the future periods in a year. Let \( D \)
and \( R \) represent, respectively, the demand and the release in a period. Let the capacity of the reservoir be \( K \). Then the standard operating policy for the period is represented as illustrated in Fig. 5.3. The available water in any period is the sum of the storage, \( S \), at the beginning of the period, and the inflow \( Q \) during the period. The release is made as per the line \( OABC \) in the figure.

Figure 5.3 implies the following:

Along \( OA \): Release = water available; reservoir will be empty after release.
Along \( AB \): Release = demand; excess water is stored in the reservoir (filling phase).
At \( A \): Reservoir is empty after release.
At \( B \): Reservoir is full after release.
Along \( BC \): Release = demand + excess of availability over the capacity (spill)

In other words, the release in any time period is equal to the availability, \( S + Q \), or demand, \( D \), whichever is less, as long as the availability does not exceed the sum of the demand and the capacity. Once the availability exceeds the sum of the demand and the capacity, the release is equal to demand plus excess available over the capacity. It is to be noted that the releases made as per the standard operating policy are not necessarily optimum as no optimization criterion is used in the release decisions. For highly stressed systems (systems in which water availability is less than the demand in most periods), the standard operating policy performs poorly in terms of distributing the deficits across the periods in a year.

With evaporation loss \( E \) included, the standard operating policy may be expressed as

\[
R_t = D_t \text{ if } S_t + Q_t - E_t \geq D_t
\]
\[
= S_t + Q_t - E_t, \text{ otherwise}
\]
\[
O_t = (S_t + Q_t - E_t - D_t) - K \text{ if positive}
\]
\[
= 0 \text{ otherwise}
\]
\[
S_{t+1} = S_t + Q_t - E_t - R_t - O_t, \text{ with } R_t \text{ and } O_t \text{ determined as above}
\]
Model Development

$S_t$ is the storage at the beginning of the period $t$, $Q_t$ is the inflow during the period $t$, $D_t$ is the demand during the period $t$, $E_t$ is the evaporation loss during the period $t$, $R_t$ is the release during the period $t$, and $O_t$ is the spill (overflow) during the period $t$. Note that $S_{t+1} = K$, if $O_t > 0$.

**Example 5.2.1** An illustrative example of the standard operating policy is shown in Table 5.3, for a reservoir with capacity $K = 350$ units and an initial storage of 200 units. Inflow $Q_t$, demand $D_t$ and evaporation $E_t$ are known values, as given in the table.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$S_t$</th>
<th>$Q_t$</th>
<th>$D_t$</th>
<th>$E_t$</th>
<th>$R_t$</th>
<th>$S_{t+1}$</th>
<th>$O_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>200.00</td>
<td>70.61</td>
<td>51.68</td>
<td>10</td>
<td>51.68</td>
<td>208.93</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>208.93</td>
<td>412.75</td>
<td>127.85</td>
<td>8</td>
<td>127.85</td>
<td>350.00</td>
<td>155.63</td>
</tr>
<tr>
<td>3</td>
<td>350.00</td>
<td>348.40</td>
<td>127.85</td>
<td>8</td>
<td>127.85</td>
<td>350.00</td>
<td>212.55</td>
</tr>
<tr>
<td>4</td>
<td>350.00</td>
<td>142.29</td>
<td>65.27</td>
<td>8</td>
<td>65.27</td>
<td>350.00</td>
<td>69.02</td>
</tr>
<tr>
<td>5</td>
<td>350.00</td>
<td>103.78</td>
<td>27.18</td>
<td>6</td>
<td>27.18</td>
<td>350.00</td>
<td>70.60</td>
</tr>
<tr>
<td>6</td>
<td>350.00</td>
<td>45.00</td>
<td>203.99</td>
<td>6</td>
<td>203.99</td>
<td>350.00</td>
<td>185.01</td>
</tr>
<tr>
<td>7</td>
<td>185.01</td>
<td>19.06</td>
<td>203.99</td>
<td>5</td>
<td>199.07</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>8</td>
<td>0.08</td>
<td>14.27</td>
<td>179.47</td>
<td>5</td>
<td>9.27</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>9</td>
<td>0.00</td>
<td>10.77</td>
<td>89.76</td>
<td>6</td>
<td>4.77</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>10</td>
<td>0.00</td>
<td>8.69</td>
<td>203.99</td>
<td>6</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>11</td>
<td>0.69</td>
<td>9.48</td>
<td>0.00</td>
<td>8</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>12</td>
<td>2.17</td>
<td>18.19</td>
<td>0.00</td>
<td>10</td>
<td>0.00</td>
<td>10.36</td>
<td>0.00</td>
</tr>
</tbody>
</table>

**5.2.2 Optimal Operating Policy Using LP**

One of the classical problems in water resources systems modelling is the derivation of an optimal operating policy for a reservoir to meet a long-term objective. Modelling techniques to be used depend on whether the reservoir inflows are treated deterministic or stochastic. In this chapter only the case of deterministic inflows is dealt with. Operating policies considering stochastic inflows are discussed in Chapter 6.

Given a reservoir of known capacity $K$, and a sequence of inflows, the reservoir operation problem involves determining the sequence of releases $R_t$ that optimizes an objective function. In general, the objective function may be a function of the storage volume and/or the release. A single, simplified reservoir system is represented as shown in Fig. 5.4.

![Fig. 5.4 | Single Reservoir Operation](image-url)
Consider the simplest objective of meeting the demand to the best extent possible (the same objective as considered in the standard operating policy), such that the sum of the demands met over a year is maximum. This may be formulated as a LP problem as follows:

\[
\text{Max } \sum R_t \quad (5.2.1)
\]

Subject to

\[
S_{t+1} = S_t + Q_t - R_t - E_t \quad \forall t \quad (5.2.2)
\]

\[
R_t \leq D_t \quad \forall t \quad (5.2.3)
\]

\[
S_t \leq K \quad \forall t \quad (5.2.4)
\]

\[
R_t \geq 0 \quad \forall t \quad (5.2.5)
\]

\[
S_t \geq 0 \quad \forall t \quad (5.2.6)
\]

\[
S_{T+1} = S_1 \quad (5.2.7)
\]

where \( T \) is the last period in the year, and all other terms are as defined in Section 5.2.1.

The constraint (5.2.3) restricts the release during a period to the corresponding demand, while the objective function (5.2.1) maximizes the sum of the releases. Thus the model aims to make the release as close to the demand as possible over the year. To ensure that the overflows \( O_t \) assume a nonzero value in the solution only when the storage at the end of the period is equal to the reservoir capacity, \( K \), integer variables may be used as discussed in Section 5.1.4. Constraint (5.2.7) makes the end of the year storage equal to the beginning of the (next) year’s storage, so that a steady state solution is achieved.

When the initial storage at the beginning of the first period is known, an additional constraint of the form, \( S_1 = S_0 \) may be included, where \( S_0 \) is the known initial storage, in which case the sequence of releases obtained would be optimal only with respect to the particular initial storage. The end of the year storage, \( S_{T+1} \), may be set equal to the known initial storage (5.2.7), if a steady state solution is desired, or may be left free (i.e. constraint 5.2.7 is excluded) if only the release sequence for one year, with known initial storage, \( S_0 \), is of interest.

**Example 5.2.2** Table 5.4 below is constructed with results obtained by solving the LP model (5.2.1) to (5.2.7) with inflow \( Q_t \), demand \( D_t \) and evaporation \( E_t \) values used in Example 5.2.1.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( S_t )</th>
<th>( Q_t )</th>
<th>( D_t )</th>
<th>( R_t )</th>
<th>( E_t )</th>
<th>( S_{t+1} )</th>
<th>( O_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10.26</td>
<td>70.61</td>
<td>51.68</td>
<td>51.68</td>
<td>10</td>
<td>19.29</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>19.29</td>
<td>412.75</td>
<td>127.85</td>
<td>127.85</td>
<td>8</td>
<td>286.19</td>
<td>0.00</td>
</tr>
<tr>
<td>3</td>
<td>296.19</td>
<td>348.40</td>
<td>127.85</td>
<td>127.85</td>
<td>8</td>
<td>350.00</td>
<td>158.74</td>
</tr>
<tr>
<td>4</td>
<td>350.00</td>
<td>142.29</td>
<td>65.27</td>
<td>65.27</td>
<td>8</td>
<td>350.00</td>
<td>69.02</td>
</tr>
<tr>
<td>5</td>
<td>350.00</td>
<td>103.78</td>
<td>27.18</td>
<td>27.18</td>
<td>6</td>
<td>350.00</td>
<td>70.60</td>
</tr>
</tbody>
</table>

(Contd.)
Model Development

<table>
<thead>
<tr>
<th>I</th>
<th>S_{t}</th>
<th>Q_{t}</th>
<th>D_{t}</th>
<th>R_{t}</th>
<th>R_{t+1}</th>
<th>S_{t+1}</th>
<th>Q_{t+1}</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>350.00</td>
<td>45.00</td>
<td>203.99</td>
<td>39.00</td>
<td>6</td>
<td>350.00</td>
<td>0.00</td>
</tr>
<tr>
<td>7</td>
<td>255.19</td>
<td>14.27</td>
<td>179.47</td>
<td>108.87</td>
<td>5</td>
<td>84.99</td>
<td>0.00</td>
</tr>
<tr>
<td>8</td>
<td>84.99</td>
<td>10.77</td>
<td>89.76</td>
<td>89.76</td>
<td>6</td>
<td>86.99</td>
<td>0.00</td>
</tr>
<tr>
<td>9</td>
<td>0.00</td>
<td>8.69</td>
<td>0.00</td>
<td>0.00</td>
<td>8</td>
<td>0.69</td>
<td>0.00</td>
</tr>
<tr>
<td>10</td>
<td>0.69</td>
<td>9.48</td>
<td>0.00</td>
<td>0.00</td>
<td>8</td>
<td>2.17</td>
<td>0.00</td>
</tr>
<tr>
<td>11</td>
<td>0.69</td>
<td>9.48</td>
<td>0.00</td>
<td>0.00</td>
<td>10</td>
<td>10.36</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Note that the values of $R_t$ in Tables 5.3 and 5.4 are not expected to be identical, as the two policies are not strictly identical.

**Rule Curves for Reservoir Operation**  Reservoir operation rules are guides for those responsible for reservoir operation. They apply to reservoirs which are to be operated in a steady state condition. A rule curve indicates the desired reservoir release or storage volume at a given time of the year. Some rules identify storage volume targets that the operator is to maintain, as far as possible, and others identify storage zones, each associated with a particular release policy. The rule curves are best derived through simulation for a specified objective, although for some simple cases, it may be possible to derive them using optimization.

In Example 5.2.2 given earlier, the end of period storages, $S_{t+1}$, define the rule curve. They are the desired end-of-period storages (target storages) to be maintained in operating the reservoir in different time periods.

The resulting rule curve is as shown in Fig. 5.5. Note that if the objective function is changed, in general, a different rule curve is obtained.

![Reservoir Rule Curve for Example 5.2.2](image)

**Multireservoir Operation**

A multireservoir system operation problem may be similarly formulated as an LP problem. Consider, for example, the three-reservoir system shown in Fig. 5.6. The system serves the purposes of water supply, flood control and hydropower generation. Release for water supply is passed through the
Reservoir Systems—Deterministic Inflow

powerhouse generating power, and losses in the powerhouse are negligible. Benefits from hydropower are expressed as a function of storage alone. \( B_i \), \( B_j \), and \( B_k \) are respectively the net benefits associated with unit release, unit available flood freeboard (= reservoir capacity – available storage), and unit storage for reservoir \( i \) in period \( t \). A portion \( \alpha_i \) and \( \alpha_j \) of the release made at reservoirs 1 and 2 respectively add to the natural inflow of reservoir 3. Further, a minimum flood storage \( F_{min}^i \) needs to be ensured during flood season at the reservoir \( i \). Maximum release that may be made at reservoir \( i \) is \( R_{max}^i \). It is assumed, in the following illustration, that

\[
B'_i = B_i \quad \forall i \\
B''_i = B''_i \quad \forall i \\
\text{and} \quad B'_i = B'_i \quad \forall i
\]

The LP problem may be stated as

\[
\text{max} \sum_{i=1}^{3} \sum_{t=1}^{T} \left[ B'_i R'_i + B''_i (K'_i - S'_i) + B'_i S'_i \right]
\]

subject to

\[
S'_{i+1} = S'_i + Q'_i - R'_i - E'_i \quad \forall t, \quad i = 1, 2 \\
S''_{i+1} = S''_i + Q''_i + \alpha_i R'_i + \alpha_j R'_j - R''_i - E''_i \quad \forall t, \quad i = 3 \\
S'_i \leq K_i \quad \forall t, \quad i = 1, 2, 3 \\
K_i - S'_i \geq F_{min}^i \quad \forall t \in \text{Flood Season} \\
R'_i \leq R_{max}^i
\]

Fig. 5.6 | Multireservoir Operation Problem

---

**Reservoir System**

Deterministic Inflow

---
In the model, the index $i$ refers to a reservoir ($i = 1, 2, 3$), and $t$ to time period ($t = 1, 2, \ldots, T$). $Q_i$ is the natural inflow, $K_i$ is reservoir capacity, $S_i$ is storage, $R_i$ is release, $E_i$ is evaporation, and $O_i$ is overflow (spill). The storage $S_i$ is the storage at the beginning of period $t$.

**Example 5.2.1** Consider the data given in the table below for a three-period, three-reservoir system. The reservoir system configuration is as shown in Fig. 5.6.

<table>
<thead>
<tr>
<th>Reservoir</th>
<th>Inflow</th>
<th>$K$</th>
<th>$B_i$</th>
<th>$R_i$</th>
<th>$E_i$</th>
<th>$O_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i=1$</td>
<td>$t=1$</td>
<td>25</td>
<td>10</td>
<td>15</td>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$t=2$</td>
<td>10</td>
<td>30</td>
<td>15</td>
<td>15</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$t=3$</td>
<td>20</td>
<td>12</td>
<td>15</td>
<td>20</td>
<td>2</td>
</tr>
<tr>
<td>$i=2$</td>
<td>$t=1$</td>
<td>10</td>
<td>30</td>
<td>15</td>
<td>15</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$t=2$</td>
<td>20</td>
<td>12</td>
<td>15</td>
<td>20</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$t=3$</td>
<td>30</td>
<td>15</td>
<td>20</td>
<td>20</td>
<td>3</td>
</tr>
<tr>
<td>$i=3$</td>
<td>$t=1$</td>
<td>20</td>
<td>12</td>
<td>15</td>
<td>20</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$t=2$</td>
<td>30</td>
<td>15</td>
<td>20</td>
<td>20</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$t=3$</td>
<td>40</td>
<td>15</td>
<td>20</td>
<td>20</td>
<td>4</td>
</tr>
</tbody>
</table>

*The benefit coefficients $B_1, B_2,$ and $B_3$ are assumed constant for all three periods.*

$\alpha_1 = 0.2$ and $\alpha_2 = 0.3$

The solution of the LP problem is as follows:

<table>
<thead>
<tr>
<th>Reservoir 1</th>
<th>$K$</th>
<th>$R$</th>
<th>$(K-S)_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t=1$</td>
<td>10</td>
<td>17</td>
<td>10</td>
</tr>
<tr>
<td>$t=2$</td>
<td>15</td>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>$t=3$</td>
<td>20</td>
<td>26</td>
<td>10</td>
</tr>
</tbody>
</table>

A general reservoir operation problem is to derive a stationary policy for a repeating sequence of inflows $\{Q_i\}$ every year, where $i$ is a ‘within-year’ period, i.e. less than one year. The stationary policy, which may be derived using DP, specifies the release as a function of the storage in a period. The objective is to derive an operating policy which results in the maximized annual net benefit in the long run (steady state).
As shown in Fig. 5.7, the computations start at some distant year in the future in the last period, \( T \). The choice of this year is such that the computations yield a steady state solution in the end. The computations are similar to those carried out for the single year policy discussed in Sec. 2.3.5 with the following modifications:

1. There are no boundary conditions. The initial (beginning of the year) or the final (end of the year) storage values are not specified, and we seek the policy for all possible storage states.

2. The computations extend beyond the one year horizon, with the stage index \( n \) continuously increasing from \( n = 1, 2, \ldots, T, T + 1, T + 2, \ldots \), and the period index, \( t \), keeping track of the position of the particular stage within the year, \( t = T, T - 1, \ldots, 1 \) (Fig. 5.7).

3. The computations are carried out until the solution reaches a steady state. The steady state is said to be reached at stage \( n \), when the annual net benefit given by \( f_n^* (S_t) = f_{n+1}^*(S_t) \) converges to a constant value, for all \( S_t \), where \( f_n^*(S_t) \) is the accumulated net benefit up to (and including) stage \( n \). This implies that the net benefit over the next \( T \) periods from stage \( n \) is a constant for all possible initial storages at that stage.

The general recursive equation is written as

\[
 f_n^*(S_t) = \max \{ B_t(S_t, R_t) + f_{n+1}^*(S_t + Q_t - R_t) \}
\]

\[0 \leq R_t \leq S_t + Q_t, \quad S_t + Q_t - R_t \leq K\]

where \( B_t(S_t, R_t) \) is the objective function value in period \( t \) for specified storage, \( S_t \), at the beginning of the period \( t \), and \( R_t \), the release during period \( t \). \( K \) is the known reservoir capacity and \( Q_t \) is the inflow during period \( t \).

**Example 2.2.2** The steady state policy is derived for the example discussed in Section 2.3.5. The first year computations are the same as the complete solution for the annual problem discussed earlier, with the addition that, in the last stage, \( n = 4 (t = 1) \), we solve for all values of \( S_t \), instead of the specified value 0. The following tables give summary results of the computations, starting from Year 1 and proceeding in the backward direction.
Model Development

<table>
<thead>
<tr>
<th>Year 1</th>
<th>Year 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 1 )</td>
<td>( n = 5 )</td>
</tr>
<tr>
<td>( t = 4 )</td>
<td>( t = 4 )</td>
</tr>
<tr>
<td>( Q_i = 2 )</td>
<td>( Q_i = 2 )</td>
</tr>
<tr>
<td>( S_i )</td>
<td>( S_i )</td>
</tr>
<tr>
<td>( f^*_i(S_i) )</td>
<td>( f^*_i(S_i) )</td>
</tr>
<tr>
<td>( R^*_i )</td>
<td>( R^*_i )</td>
</tr>
<tr>
<td>0   320 2 0   1780 1, 2</td>
<td></td>
</tr>
<tr>
<td>1   480 3 1   1940 1, 3</td>
<td></td>
</tr>
<tr>
<td>2   520 4 2   2010 1, 2, 3</td>
<td></td>
</tr>
<tr>
<td>3   520 4, 5 3 2170 1, 3</td>
<td></td>
</tr>
<tr>
<td>4   520 4, 5 4 2240 2, 3</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Year 1 (Contd.)</th>
<th>Year 2 (Contd.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 2 )</td>
<td>( n = 6 )</td>
</tr>
<tr>
<td>( t = 3 )</td>
<td>( t = 3 )</td>
</tr>
<tr>
<td>( Q_i = 3 )</td>
<td>( Q_i = 3 )</td>
</tr>
<tr>
<td>( S_i )</td>
<td>( S_i )</td>
</tr>
<tr>
<td>( f^*_i(S_i) )</td>
<td>( f^*_i(S_i) )</td>
</tr>
<tr>
<td>( R^*_i )</td>
<td>( R^*_i )</td>
</tr>
<tr>
<td>0   800 2, 3 0 2260 1, 2, 3</td>
<td></td>
</tr>
<tr>
<td>1   960 3 1 2420 1, 3</td>
<td></td>
</tr>
<tr>
<td>2   1000 3, 4 2 2490 1, 2, 3</td>
<td></td>
</tr>
<tr>
<td>3   1040 4 3 2560 2, 3</td>
<td></td>
</tr>
<tr>
<td>4   1040 4, 5 4 2720 3</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Year 3</th>
<th>Year 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 3 )</td>
<td>( n = 8 )</td>
</tr>
<tr>
<td>( t = 2 )</td>
<td>( t = 2 )</td>
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Five cycles of computations are shown here. In this example, the solution reached a steady state at Stage 9. The attainment of steady state may be verified by observing that for a given time period, the optimal release $R^*$ is constant across consecutive years for a specified initial storage. For example, the $R^*$ corresponding to $S_t = 1$ in period $t = 1$, $n = 20$, in Year 5, which is 1, 2, or 3, is the same as $R^*$ for $S_t = 1$ in period $t = 1$, $n = 16$, in Year 4. Further, it may also be verified that the value, $(f^{(n+1)}(S_t) - f^{(n)}(S_t))$, converges to a constant value when the steady state is reached.

For $t = 1$, $S_t = 1$, between year 5 and year 4

$$f^{(n+1)}(S_t) - f^{(n)}(S_t) = f^{(16)}(1) - f^{(16)}(1) = 7370 - 5910 = 1460.$$  
Similarly, between Year 4 and Year 3, we see that, for $t = 1$

$$f^{(n+1)}(S_t) - f^{(n)}(S_t) = f^{(12)}(1) - f^{(12)}(1) = 5910 - 4450 = 1460,$$  
which is the same as the one obtained for Year 5 and Year 4.

It may also be verified from the tables that this annual benefit is independent of the storage state. That is, irrespective of the storage $S_t$ used in the annual benefit term, $(f^{(n+1)}(S_t) - f^{(n)}(S_t))$, the value remains the same (1460) once the steady state is reached. In this particular example, the steady state was reached right in the third year, and there was no need to carry out computations further. Note that the annual benefit of 1460 is also the value of the optimal objective function we obtained when we solved the problem for only one year with the initial storage specified (Sec. 2.3.5), because the initial storage specified for that problem turns out to be the steady state storage for the period, (this may be verified by simulating the reservoir operation with the steady state policy starting with any arbitrary storage, but with the same deterministic inflows repeating every year; after a few cycles of simulation the storage in each period reaches a steady state value), and the optimal release sequence also corresponds to the steady state release sequence. However, this coincidence will not occur in a general case. In general, the steady state may be expected to be reached within about 4 or 5 cycles. Once the steady state is reached, the stationary policy is specified as $R^*(S_t)$, which gives the release from the reservoir for a given storage in period $t$. Note that there are multiple solutions for several storage states (e.g. all storage states in period 4), and any of the solutions will yield the same optimal value of the objective function.

5.2.4 Simulation of Reservoir Operation for Hydropower Generation

Hydropower Generation: Preliminary Concepts

The kinetic energy produced by 1 m$^3$ of water falling through a distance of 1 m is equal to $\rho g H = 1000 \times 9.81 \times 1 = 9810$ Nm. Here $\rho$ is the density of water (1000 kg/m$^3$), $g$ is the acceleration due to gravity (9.81 m/s$^2$), and $H$ is the
height (head) in metres from which the water falls. The energy generated per second, called the power, is 9810 watts. By definition, then, a discharge of 1 \( \text{m}^3/\text{s} \), produces 9810 watts of power at a head of 1 m. Hence, an average flow of \( q \), \( \text{m}^3/\text{s} \), falling through a height of \( H \), metres continuously in a period \( t \) (e.g. a week or a month), will yield a power of 9810 \( q \cdot H \) watts, or 9.81 \( q \cdot H \) kilowatts (kW). A useful unit for energy is the kilowatt-hour (kWh). It may be verified that the total kilowatt-hours of energy produced in period \( t \) will be

\[
\text{kWh}_t = 9.81 \times 10^6 \frac{R \cdot H}{3600}
\]

where \( R \) is the total flow in \( \text{Mm}^3/\text{t} \), and \( H \) is the head in metres. This expression assumes 100 per cent conversion of energy. Considering an overall efficiency, \( \eta \), the energy generated in kWh during the period \( t \), is written as

\[
\text{kWh}_t = 2725 \frac{R \cdot H}{3600} \eta
\]

**Firm Power and Secondary Power**

The amount of power that can be generated with certainty without interruption at a site, is called the firm power, and the corresponding energy is called the firm energy. The firm power can be thought of as the power produced at a site with 100% reliability all the time (that is, at no time the power produced will be less than the firm power). The power that can be generated more than 50% of time is called the secondary power. Secondary power is thus less reliable than firm power and, therefore, is generally priced lower than firm power.

The run-of-the-river power plants are those which produce power by using water directly from the stream without any requirements of storage. A natural drop (or head) exists in such situations, which is made use of, for producing power. The power generated from the run-of-the-river schemes will be quite small compared to that generated from storage reservoirs. For the run-of-the-river plants, the head available is nearly constant throughout the year and, therefore, the flow rate itself determines the power generated. The firm power, thus, corresponds to the minimum flow at that site (also referred to as the firm yield of the river).

For example, consider a river with a minimum monthly flow of 20 \( \text{Mm}^3/\text{t} \). If a drop of 30 m is available at a site on the river, the firm energy that can be produced at the site in a month, with an efficiency of 0.7, is simply,

\[
2725 \frac{R \cdot H}{3600} \eta = 2725 \times 20 \times 30 \times 0.7 = 1144500 \text{ kWh} = 1.1445 \text{ GWh} \text{ (giga watt hour)}
\]

For determining the secondary power for a run-of-the-river plant, we must know the flow with 50% reliability (i.e. the flow which will be equalled or exceeded 50% of the time), and substitute that flow magnitude for \( R \), in the above expression.

**Reservoir Operation for Hydropower Generation**

Most major hydropower plants accompany a storage reservoir from where water is drawn to produce power. The storage reservoir, apart from regulating the flow, also ensures that adequate head is available for generating the power.
Unlike in the case of run-of-the-river schemes, the head, flow, and storage are all interdependent in the case of a power plant that has a storage reservoir. For obtaining the firm energy at a reservoir, the flows must be routed through the reservoir, simulating the energy production from period to period. The data required for such an exercise are:
1. The inflow series at the reservoir,
2. The storage–elevation–area relationships for the reservoir, and
3. The power plant efficiency.

The procedure involves, essentially, applying the reservoir storage continuity and simulating the power generation. Figure 5.8 shows the components of a hydropower system with a storage reservoir.

\[
\text{Net Head for Power Generation} = H_{1} - H_{2}
\]

The simulation procedure to produce the desired firm power \( P \) is illustrated below.

From the expression, \( \text{kWh}_t = 2725 \ R_t \ H_t \ \eta \), we write, for a period of one month (30 days),

\[
\text{Power in MW} = 2725 \ R_t \ H_t \ \eta/(1000 \times 30 \times 24)
\]

and thus,

\[
R_t = P/(0.003785 \ H_t \ \eta)
\]

where \( R_t \) is the release into penstocks in Mm\(^3\), \( P \) is the power in MW, \( H_t \) is the net head in metres and \( \eta \) is the plant efficiency.

**Reservoir Storage Continuity** The reservoir storage continuity is considered in the simulation as

\[
S_{t+1} = S_t + Q_t - R_t - E_t - \text{spill}_t
\]

where \( S_t \) is the storage at the beginning of the month \( t \) (\( t = 1, 2, \ldots, 12 \)), \( Q_t \) is the reservoir inflow during the month \( t \), \( R_t \) is the power release required in month \( t \) to generate the specified power corresponding to the head, resulting from the average of storages \( S_t \) and \( S_{t+1} \), \( E_t \) is the evaporation loss in month \( t \), corresponding to the water spread area at the average storage \( (S_t + S_{t+1})/2 \), and \( \text{spill}_t \) is the spill during period \( t \). Since the head available for generating power and the evaporation loss both depend on the storage at the beginning and the end of the period, these values are obtained through an iterative procedure, assuming a value for the average storage and correcting it iteratively till convergence occurs. This is done as follows for a known value of \( S_t \).
1. Assume average storage $S_t = S_t$
2. Obtain net head, $H_t$ and water spread area, $A_t$, corresponding to $S_t$. These are got from the storage-area-elevation relationships for the reservoir.
3. Determine the release, $R_t$, required for generating the specified power, $P$ (in MW), from:
   \[ R_t = P/(0.003785 \times H_t) \]
4. The evaporation loss is got from $E_t = A_t \times e_t$, where $e_t$ is the rate of evaporation (depth) in period $t$, and $A_t$ corresponds to the storage $S_t$.
5. Get the end of period storage,
   \[ S_{t+1} = S_t + Q_t - R_t - E_t \]
   if $S_{t+1} <$ reservoir capacity, $K$.
   \[ = K, \text{ otherwise} \]
6. Get the average storage, $\bar{S}_t = (S_t + S_{t+1})/2$
7. If $\bar{S}_t$ is nearly equal to $S_t$, the computed values of $H_t, R_t, E_t$, and $S_{t+1}$ are acceptable. Else, set $\bar{S}_t = S_t$ and go to step 2; repeat steps 2 to 7 until the computed values of $H_t, R_t, E_t$, and $S_{t+1}$ are acceptable.
   This procedure converges quickly, and is very useful in simulation of reservoir operation for hydropower generation.

Example 5.2.5 Simulate the reservoir operation for hydropower generation with the following data:
Reservoir capacity = 1226 Mm$^3$; minimum power desired in a month = 73.5 MW. The storage-elevation data for the site is as given in Table 5.5, with an allowance of 47 m (R.L) for the tail race water level. Inflows are as shown in the Table 5.6. Rate of evaporation for the 12 months starting June are: 11, 9, 8, 9, 8, 7, 8, 10, 13, 14, and 11 cms. The plant efficiency is 81.54%. Initial storage = 824.63 Mm$^3$. The spill produces additional power with the head equal to maximum head.

Table 5.5 Elevation–Capacity–Area Values

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<th>Capacity (Mm$^3$)</th>
<th>Area (Mm$^2$)</th>
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</table>
Solution:

With \( \eta = 81.54\% \), the power release (draft) required to generate a known power \( P(\text{MW}) \) during a month is given, for this example, by

\[
R = \frac{P}{(0.0030864 H)}
\]

(5.2.8)

In simulation, the power draft \( R_t \) is determined as the lower value of the quantity given by Eq. (5.2.8), and the water available in the month \( t \). That is,

\[
R_t = \frac{P}{(0.0030864 H_t)} \quad \text{if } S_t + Q_t - E_t \geq \frac{P}{(0.0030864 H_t)}
\]

\[
= S_t + Q_t - E_t \quad \text{otherwise}
\]

The net head \( H_t \), power draft \( R_t \), evaporation loss, \( E_t \), and end-of-the-period storage, \( S_{t+1} \), are all determined simultaneously by the iterative procedure discussed before. A linear interpolation is used to get the net head and water spread area for a given capacity. The tail water elevation of 47 m is deducted from the head obtained from the elevation–capacity–area relationship, to obtain the net head.

**Primary and Additional Power**

Whenever the power draft is adequate to generate the specified power \( P = 73.5 \text{ MW} \), the primary power is equal to \( P \) itself. When the power draft is less than that required to generate the power \( P \), the primary power \( P' \) is given by

\[
P' = 0.0030864 R_t H_t
\]

On the other hand, the additional power is generated only when the reservoir spills. The spill during a month is computed based on the end-of-the-period storage, as

\[
\text{spill}_t = 0 \quad \text{if } S_{t+1} \leq K
\]

\[
= S_{t+1} - K \quad \text{otherwise}
\]

where \( \text{spill}_t \) is the spill during period \( t \) and \( K \) is the reservoir capacity.

When there is a spill during a period, the end-of-the-period storage, \( S_{t+1} \), is set to \( K \), after computing the spill. The additional power is computed based on the spill with a net head corresponding to the full reservoir level, as

\[
P'' = 0.0030864 \text{spill}_t H_{\text{max}}
\]

where \( H_{\text{max}} \) is the net head corresponding to the full reservoir level.

The reservoir operation is simulated thus and the power generated in any month is obtained. Table 5.6 shows the simulation results for one sample year for which inflows are given. With a large number of years of data, this procedure may be used to obtain the reliability of power generation at a site. We may also use the procedure to estimate the minimum required reservoir capacity for generating a known amount of power at a given reliability by carrying out the simulation with various capacities and obtaining a capacity–reliability relationship.
### Problems

**Note:** In the Problems 5.2.1 to 5.2.3, draw a schematic diagram of the system, define notations and write the mathematical formulation.

#### 5.2.1
A reservoir of live capacity $K$ irrigates a total area $A$ in which five crops may be grown. For each unit of water supplied to a unit area of crop $i$ ($i = 1, 2, \ldots, 5$), a benefit of $W_i^t$ is obtained in period $t$ ($t = 1, 2, \ldots, 12$). Each crop must be grown on at least 12% of the total area. In addition, the reservoir also serves the purpose of navigation, for which a constant release of $R_N$ is made during each period. Assuming the inflow $Q_t$ to be known for each period $t$, formulate an optimization model to decide the optimal cropping pattern (i.e. area $A_i$ on which crop $i$ should be grown) and the release policy (i.e. release $R_t$ from the reservoir in period $t$ for irrigation) to maximize the benefits. Is the problem so formulated an LP problem? If not, why not?

#### 5.2.2
A three-reservoir system in a river basin functions to absorb floods during the wet season ($t = 1, 2, \ldots, 5$) and to provide irrigation during the dry season ($t = 6, 7, \ldots, 12$). Reservoirs 1 and 2 and reservoirs 3 and 2 are in series with reservoir 2 being the downstream reservoir in both cases. The three reservoirs together serve an irrigation area in the dry season ($t = 6, 7, \ldots, 12$). At the beginning of the period, $t = 6$, all the reservoirs are full and there is no natural inflow to any reservoir during the dry season. Water may be released for irrigation from reservoirs 2 and 3. The irrigation demand during period $t$ is known as $D_t$ ($t = 6, 7, \ldots, 12$). Each unit of water released from reservoir $i$ ($i = 2, 3$) brings a benefit of $B_t$ in period $t$ ($t = 6, 7, \ldots, 12$). Formulate an LP problem to obtain the release policy for each reservoir for maximizing the benefits. The capacities of the reservoirs are known.

### Table 5.6 | Simulation Results (One Year)

<table>
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<tr>
<th>Month</th>
<th>Storage</th>
<th>Inflow</th>
<th>Evap</th>
<th>Release</th>
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*Additional power is produced only when spill occurs.*
The irrigation releases from the reservoir \( i \) \((i = 2, 3)\) are restricted by the canal capacity \( C_i \). Assume that the total benefits from flood control up to the beginning of period 6 is a constant, and that no release is necessary from any reservoir to absorb floods. [Hint: Formulate the LP problem for the dry season only.]

5.2.3 A water resource system consists of 4 reservoirs. Reservoirs 1, 2, and 4 are in series (in that order, with 1 being the first (topmost) upstream reservoir). Reservoirs 3 and 4 are also in series, 3 being the upstream reservoir. Each of the reservoirs 1, 2, and 3 serves its own powerhouse. The release made into the powerhouse subsequently reaches the immediate downstream reservoir. Each unit of power generated brings a benefit of \( B_t \) in period \( t \). The power generated at the reservoir \( i \) is limited by the plant capacity \( p_i \). The storage–elevation relationship for the reservoir \( i \) \((i = 1, 2, 3)\) may be assumed to be linear (known). In addition, reservoir 4 serves an irrigation area. Each unit of release made for irrigation from reservoir 4 fetches a benefit of \( W_t \) in period \( t \). Assuming the inflows and capacities of reservoirs to be known, formulate an optimization model to maximize the benefits from the system. Identify the decision variables in the problem. For a 12-period problem, how many constraints and how many decision variables will result from the formulation?

5.2.4 Re-solve Example 5.2.1 with reservoir capacities of 300, 400, 450, and 500 units using the same values of \( Q_t, D_t, \) and \( E_t \). Obtain the annual deficit in release, \( \sum (R_t - D_t) \), for each capacity to examine if the annual deficit may be reduced to zero by increasing the capacity. Also carry out simulations to examine the effect of initial storage on the annual deficit at the beginning of the first period for each of the capacities (based on one-year simulations).

5.2.5 A reservoir has a live capacity of 870 Mm\(^3\). When possible, a constant release of 150 Mm\(^3\) is made from the reservoir during all periods for power generation. If this release is not possible, all available water in the reservoir is released for power generation. The head available for power generation is given by \( h = 30 + (S/18) \), where \( h \) is the head in metres, and \( S \) is the reservoir storage in Mm\(^3\). Assuming an initial storage of 500 Mm\(^3\), simulate the power generation from the reservoir. The inflow during the six periods in a year is 75, 100, 540, 328, 0, and 0. Assume that overflows are not available for power generation. Neglect losses.

5.2.6 Two reservoirs are in series, with Reservoir 1 being on the upstream. Reservoir 1 has a capacity of 1172 Mm\(^3\) and Reservoir 2 has a capacity of 2647 Mm\(^3\). Both reservoirs serve their individual irrigation areas. Assuming that there is a slightly higher weightage of 1.05 attached to the demands from Reservoir 2, formulate and solve an LP problem to maximize the sum of weighted demands from the two reservoirs.
Reservoir Systems—Deterministic Inflow

Reservoirs. The monthly inflows (Mm$^3$) at the two reservoirs are given below:

- **Res 1**: 155 1354 1087 395 293 92 56 19 14 6 12 36
- **Res 2**: 121 399 614 411 329 222 146 66 43 45 65 95

40% of release made from Reservoir 1 subsequently joins as return flows to Reservoir 2, in addition to the inflows shown above.

REFERENCES


Further Reading

Decisions relating to most water resources systems need to be made in the face of hydrologic uncertainty. The hydrologic variables such as rainfall in a command area, inflow to a reservoir, evapotranspiration of crops which influence decision making in water resources, are all random variables. Optimization models developed for water resources management must therefore be formulated to give optimal decisions with an indication of the associated hydrologic uncertainty. Two classical approaches to deal with the hydrologic uncertainty in optimization models are (a) The implicit stochastic optimization (ISO) and (b) The explicit stochastic optimization (ESO). In the ISO, the hydrologic uncertainty is implicitly incorporated. The optimization model itself is a deterministic model, in which the hydrologic inputs are varied with a number of equiprobable sequences and the deterministic optimization model is run once with each of the input sequences. The output set is then statistically analyzed to generate a set of optimal decisions. In the ESO, however, the stochastic nature of the inputs is explicitly included in the optimization model through their probability distributions. The optimization model is a stochastic model and a single run of the model specifies the optimal decisions. In this chapter, we discuss two commonly used ESO techniques: Chance Constrained Linear Programming (CCLP), and Stochastic Dynamic Programming (SDP), both of which require a basic background of probability theory. We begin with a review of the basics of probability theory.

6.1 REVIEW OF BASIC PROBABILITY THEORY

Random Variable A variable whose value is not known or cannot be measured with certainty (or is nondeterministic) is called a random variable. Examples of random variables of interest in water resources are rainfall, streamflow, time between hydrologic events (e.g. floods of a given magnitude), evaporation from a reservoir, groundwater levels, re-aeration and de-oxygenation rates, and
so on. It must be noted that any function of a random variable is also a random variable (r.v). As a convention, we use an upper case letter to denote a random variable and the corresponding lower case letter to denote the value that it takes. For example, daily rainfall may be denoted as $X$. The value it takes on a particular day is denoted as $x$. We then associate probabilities with events such as $X \geq x$, $0 \leq X \leq x$, etc.

**Discrete and Continuous Random Variables**

If an r.v. $X$ can take on only discrete values $x_1, x_2, x_3, \ldots$, then $X$ is a discrete random variable. An example of a discrete random variable is the number of rainy days in a year which may take on values such as, 10, 20, 30, …, A discrete random variable can assume a finite (or countably infinite) number of values. If an r.v. $X$ can take on all real values in a range, then it is a continuous variable. Most variables in hydrology are continuous random variables. The number of values that a continuous random variable can assume is infinite.

**Probability Distributions**

For a discrete random variable, there are spikes of probability associated with the values that the random variable assumes. Figure 6.1 is a typical plot of the distributions of probabilities associated with a discrete random variable. In case of discrete random variables, the probability distribution is called a **probability mass function** and in case of continuous random variables it is called a **probability density function**. The cumulative distribution function, $F(x)$, represents the probability that $X$ is less than or equal to $x$, and is shown in Fig. 6.2 for a discrete r.v. i.e. $F(x_k) = P(X \leq x_k)$

![Fig. 6.1 Probability Mass Function](image)

Probability distributions of continuous random variables are smooth curves. The probability density function (pdf) of a continuous random variable is denoted by $f(x)$. The cumulative distribution function (cdf) of a continuous random variable is denoted by $F(x)$. It is a nondecreasing function with a maximum value of 1. The cdf represents the probability that $X$ is less than or equal to $x$, i.e. $F(x) = P(X \leq x)$. 


Any function \( f(x) \) defined on the real line can be a valid probability density function if and only if
1. \( f(x) \geq 0 \) for all \( x \), and
2. \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \)

The pdf and the cdf are related by
\[
F(x) = \int_{-\infty}^{x} f(x) \, dx
\]

Figures 6.3 and 6.4 show typical examples of pdf and cdf respectively. For a continuous random variable, probability of the random variable taking a value exactly equal to a given value is zero, because
\[ P(X = d) = P(d \leq X \leq d) = \int_d^d f(x) \, dx = 0 \]

From Fig. 6.3
- Area under the curve to the left of \( x = a \) is \( \text{Prob}(X \leq a) \)
- Area under the curve to the left of \( x = b \) is \( \text{Prob}(X \leq b) \)
- Area between \( x = a \) and \( x = b \) is \( \text{Prob}(a \leq X \leq b) \)

For any given \( \alpha, 0 \leq \alpha \leq 1 \), we may determine a value \( x \) from the cumulative distribution such that \( F(x) = \alpha \). We then denote, \( x = F^{-1}(\alpha) \). This is illustrated in Fig. 6.5.

**Fig. 6.5 | CDF of X**

The following definitions will be useful:

*Expected value of \( X \), \( E(X) \):
\[ E(X) = \sum_x x \cdot p(x) \quad \text{for discrete r.v.s} \]
where \( p(x) \) is \( \text{Prob}(X = x) \);
\[ E(X) = \int_{-\infty}^{\infty} x \cdot f(x) \, dx \quad \text{for continuous r.v.s} \]

where \( f(x) \) is the pdf of the r.v. \( X \).

The *mean* of an r.v. \( X \), denoted as \( \mu \), is equal to the expected value, i.e.
\[ \mu = E(X) \]

If \( g(X) \) is a function of \( X \), then,
\[ E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f(x) \, dx \quad \text{... \( X \) continuous} \]
\[ = \sum_x g(x) \cdot p(x) \quad \text{... \( X \) discrete} \]

*The variance*, \( \sigma^2 \), [also denoted as \( \text{Var}(X) \)]
\[ \text{Var}(X) = \sigma^2 = E(X - \mu)^2 \]
\[ = \sum_x (x - \mu)^2 \cdot p(x) \quad \text{... \( X \) discrete} \]
\[ = \int_{-\infty}^{\infty} (x - \mu)^2 \cdot f(x) \, dx \quad \text{... \( X \) continuous} \]
Model Development

Standard deviation, $\sigma = \sqrt{\text{variance}}$ (where variance is a positive square root of variance).

Coefficient of variation, $C_v = \frac{\sigma}{\mu}$

When we have a sample of observations, $x_1, x_2, \ldots, x_n$ on the r.v. $X$, the mean, variance, and coefficient of variation may be estimated as:

Sample estimate of mean = Arithmetic average,

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

where $n$ is the number of observations, and $\bar{x}$ is the sample estimate of mean, $\mu$.

The sample estimate of variance is:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

where $S$ is the sample (unbiased) estimate of the standard deviation, $\sigma$.

Sample estimate of coefficient of variation

$$C_v = \frac{S}{\bar{x}}$$

**Example 6.1.1** A random variable, $X$ is described by a probability density function

$$f(x) = \begin{cases} 3x^2, & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find:

1. the cumulative distribution function
2. $E(X)$
3. $\text{Var}(X)$
4. $P[X \geq 0.6]$

**Solution:**

1. Cumulative distribution function,

$$F(x) = \int_{-\infty}^{x} f(x') \, dx'$$

$$= \int_{0}^{x} 3x^2 \, dx$$

$$= x^3 \quad \text{for } 0 \leq x \leq 1$$
2. $E(X) = \int_{-\infty}^{\infty} x f(x) \, dx$

$$E(X) = \int_{0}^{\infty} 3x^2 \, dx = 3/4$$

3. $\text{Var}(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) \, dx$

$$\text{Var}(X) = \int_{0}^{\infty} (x-(3/4))^2 3x^2 \, dx$$

$$= 3x^3 + \frac{27x^5}{48} + \frac{18x^4}{16} - \frac{3}{80} = 0.0375$$

4. $P[X \geq 0.6] = 1 - P[X \leq 0.6]$

$$= 1 - F(0.6)$$

$$= 1 - (0.6)^3 = 0.784$$

Example 6.1.2 Given $f(x) = 3e^{-3x}$, $x > 0$, what is the value of $x$ such that

1. $P[X \leq x] = 0.50$
2. $P[X \leq x] = 0.75$

Solution:

$$F(x) = P[X \leq x] = \int_{0}^{x} 3e^{-3x} \, dx$$

$$= 1 - e^{-3x}$$

1. $P[X \leq x] = 0.50$

$$\therefore 1 - e^{-3x} = 0.5$$

$$x = 0.231$$

2. $P[X \geq x] = 0.75$

i.e. $1 - P[X \leq x] = 0.75$

$$1 - [1 - e^{-3x}] = 0.75$$

$$e^{-3x} = 0.75$$

$$-3x = \ln 0.75$$

$$x = 0.0958$$

Normal Distribution A number of commonly used probability distribution in hydrology may be found in standard textbooks on hydrology (e.g. Haan, 1977; Chow et al., 1988). Here we discuss only three such distributions, the normal, lognormal and the exponential distributions.

The pdf of the normal distribution is given by

$$f(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty \leq x \leq +\infty$$
The mean, \( \mu \), and the standard deviation, \( \sigma \), are the only two parameters of the normal distribution. The sample estimates of \( \mu \) and \( \sigma \) are arithmetic mean, \( \bar{\mu} \), and the standard deviation, \( S \), given by:

\[
\bar{\mu} = \frac{\sum_{i=1}^{n} x_i}{n}
\]

\[
S = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{\mu})^2}{n-1}}
\]

The \( n - 1 \) in the denominator (instead of \( n \)) for \( S \) is used to get an unbiased estimate of \( \sigma \) from the sample.

Figures 6.6 and 6.7 show examples of the pdf and cdf of a normal distribution. The normal pdf is symmetrical about \( X = \mu \). We denote as \( X \sim N(\mu, \sigma^2) \) to convey that the random variable \( X \) follows normal distribution with mean \( \mu \) and variance \( \sigma^2 \).
Consider the CDF of the normal distribution 
\[ F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx \quad \text{for} \quad -\infty \leq x \leq \infty \]

Since exact integration of the expression on the right-hand side (rhs) is not possible, numerical integration is necessary to compute \( F(x) \). The inconvenience of using numerical integration separately for various values of \( \mu \) and \( \sigma \) is avoided by defining the standard normal variate, \( Z \), as 
\[ Z = \frac{X - \mu}{\sigma} \]

With this transformation, the random variable, \( Z \), follows normal distribution with mean 0 and variance 1, i.e., \( Z \sim N(0, 1) \).

The pdf of \( Z \) is denoted as \( \phi(z) \) and is given by 
\[ \phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \text{for} \quad -\infty \leq z \leq +\infty \]

The cdf of \( Z \) is 
\[ \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} \, dt \quad \text{for} \quad -\infty \leq z \leq +\infty \]

The pdf of \( Z \) is symmetrical about \( z = 0 \). Values of \( \phi(z) \) obtained for a range of \( z \) values by numerical integration are tabulated (Table 6.1). These values may be used in the computations for normal distribution. The following example illustrates the use of the standard normal distribution table.

**Example 6.1.1** The monthly streamflow at a reservoir site is represented by a random variable \( X \) which follows normal distribution with a mean of 30 units and a standard deviation of 15 units.

Find
1. \( P(X \geq 45) \)
2. \( P(X \leq 20) \)
3. The flow value which will be exceeded with a probability of 0.9.

**Solution:**
1. \( P(X \geq 45) = P(|X - \mu|/\sigma \geq (45 - 30)/15) \)
   \[ = P(Z \geq 1) \]
   \[ = P(Z \leq -1) \]
   \[ = 1 - P(Z \leq 1) \]
   \[ = 1 - 0.8413 \quad \text{from Table 6.1} \]
   \[ = 0.1587 \]
2. \( P(X \leq 20) = P(|X - \mu|/\sigma \leq (20 - 30)/15) \)
   \[ = P(|Z| \leq -10/15) \]
   \[ = P(Z \leq -0.67) \]
   \[ = 0.2486 \quad \text{(from Table 6.1)} \]
   \[ = 0.2514 \]
3. Flow value exceeded with probability 0.9
   \[ = P(Z \geq 1.28) \]
   \[ = 1 - P(Z \leq 1.28) \]
   \[ = 1 - 0.9000 \]
   \[ = 0.1000 \]
3. The problem is to find \( x \) such that \( P[X \geq x] = 0.9 \)
\[ P[X \geq x] = 0.9 \]
\[ P[Z \geq ((x - \mu)/\sigma)] = 0.9 \]
\[ \text{i.e.} \quad 1 - P[Z \leq z] = 0.9 \]
where \( z = (x - 30)/15 \)
\[ \text{i.e.} \quad P[Z \leq z] = 0.1 \quad \ldots \text{(A)} \]

In Table 6.1, we look for the value of \( z \) which gives an area of 0.1 under the standard normal distribution (see Fig. 6.8).

![Fig. 6.8 Use of Standard Normal Distribution](image)

From Fig. 6.8, it is clear that we first look for a value \( z' \) corresponding to an area of 0.4 on the right half of the standard normal curve. From Table 6.1, we get \( z' = 1.28 \).

Thus, in Eq. (A), we use \( z = -z' = -1.28 \), so that the area to the left of \(-z'\) is 0.1.
\[ (x - 30)/15 = -1.28 \]
\[ \text{or} \quad x = 10.8 \text{ units.} \]

**Lognormal and Exponential Distributions**

A random variable \( X \) follows lognormal distribution if the transformed variable \( Y = \ln(X) \) follows normal distribution. The pdf of the lognormal distribution is given by

\[ f(x) = \frac{1}{\sqrt{2\pi} \sigma_x} e^{-\frac{(\ln x - \mu)^2}{2\sigma_x^2}} \quad x > 0 \]

where \( \mu \) and \( \sigma_x^2 \) are estimated by \( \bar{y} \) and \( S_y \), with \( y = \ln x \). When a sample of values of the lognormal random variable \( X \) are available, we may thus use the transformation, \( Y = \ln X \), and work with normal distribution for \( Y \). When we have only sample estimates of the mean \( \bar{y} \) and the standard deviation, \( S_y \), of \( X \), these may be transformed to the corresponding statistics of \( Y \), using the following (Haan, 1977).
Table 6.1 Standard Normal Distribution

The table entries are the probabilities \( p \) for which \( P(0 \leq Z \leq z) \), where \( z \) ranges from 0.00 to 3.99.

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\[
\bar{y} = \frac{1}{2} \ln \left( \frac{\bar{Y}/(1 + C_v^2)}{S_y^2} \right)
\]

where \( \bar{y} \) and \( S_y \) are the sample estimates of mean and standard deviation of \( Y \), and \( C_v \) is the sample coefficient of variation of \( X \), given by \( C_v = S_x/\bar{x} \).

The exponential distribution has a pdf of \( f(x) = \lambda e^{-\lambda x}; \ x > 0, \ \lambda > 0 \). \( \lambda \) is estimated as \( \lambda = 1/\mu \), where \( \mu \) is the mean of the r.v. \( X \).

**Example 6.1.4** Annual peakflows at a location are known to be exponentially distributed with a mean of 1200 Mm\(^3\). Find the peak flow which has an exceedance probability of 0.7.

**Solution:**
Representing the peak flow at the site by a random variable \( X \), we determine the value \( x \) such that \( P[X \geq x] = 0.7 \).

The pdf of the exponential distribution is given by,
\[
f(x) = \lambda e^{-\lambda x}; \ x > 0, \ \lambda > 0 \]
where the parameter \( \lambda \) is estimated as \( \lambda = 1/\mu \), where \( \mu \) = mean.

\[
\therefore \ P[X \leq x] = F(x) = \int_{0}^{x} \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x}
\]
\[
\therefore \ P[X \geq x] = 1 - P[X \leq x] = e^{-\lambda x}
\]

In this example,
\[
P[X \geq x] = 0.7
\]
\[
e^{-\lambda x} = 0.7
\]
\[
-\lambda x = \ln 0.7
\]
\[
(-1/1200)x = -0.357
\]
i.e.
\[
x = 428.4 \text{ Mm}\(^3\).
\]

### 6.2 CHANCE CONSTRAINED LINEAR PROGRAMMING

The inflow to a reservoir is the most important random variable that introduces uncertainty in reservoir planning and operation problems. In Section 5.1, a deterministic LP model was formulated to obtain the reservoir capacity to meet a specified demand, \( D_t \) during period \( t \), when inflow \( Q_t \) and evaporation \( E_t \) during period \( t \) are known. The deterministic model is rewritten here as

Min \( K \)

subject to

\[
S_t + Q_t - R_t - E_t = S_{t+1}
\]
\[
R_t \geq D_t
\]
\[
R_t \leq R_{\text{max}}
\]
where $K$ is the reservoir capacity, $S_t$ is the storage at the beginning of period $t$, $R_t$ is the release during period $t$, $R_{max}$ is the maximum release that can be made in period $t$ (normally decided by canal capacities), and $S_{min}$ is the minimum storage below which no release is made.

In this model, the demands from the reservoir are met 100% of the time, in a feasible solution, provided all variables are deterministic. The inflow $Q_t$, however, is a random variable and thus is not known with certainty. Its probability distribution, however, may be estimated from the historical sequence of inflows. Being functions of the random variable $Q_t$, the storage $S_t$ and the release $R_t$ are also random variables.

In a constraint containing two random variables, if the probability distribution of one is known, the probabilistic behavior of the second can be expressed as a measure of chance in terms of the probability of the first variable. If a constraint contains more than two random variables, we get into computational complications, and we need to understand the specific problem clearly to reformulate the problem, if necessary, and avoid those complications.

**Chance Constraint**

The constraint, relating the release, $R_t$ (random) and demand, $D_t$ (deterministic), is expressed as a chance constraint, $P[R_t \geq D_t] \geq \alpha_t$; meaning that the probability of release equaling or exceeding the known demand is at least equal to $\alpha_t$, which is referred to as the reliability level. The interpretation of this chance constraint is simply that the reliability of meeting the demand in period $t$ is at least $\alpha_t$.

Similarly, the maximum release and the maximum and minimum storage constraints are written as

\[ P[R_t \leq R_{max}] \geq \alpha_2 \]  
\[ P[S_t \leq K] \geq \alpha_1 \]  
\[ P[S_t \geq S_{min}] \geq \alpha_4 \]

To use the constraints (6.2.1) to (6.2.3) in an optimization algorithm, we must first determine the probability distribution of $R_t$ and $S_t$ from the known probability distribution of $Q_t$. However, because $S_t$, $Q_t$, and $R_t$ are all interdependent through the continuity equation, it is, in general, not possible to derive the probability distributions of both $S_t$ and $R_t$. To overcome this difficulty and to enable the use of linear programming in the solution, a linear decision rule is appropriately defined.

**6.2.1 Linear Decision Rule**

The linear decision rule (LDR) relates the release, $R_t$, from the reservoir as a linear function of the water available in period $t$. The simplest form of such an LDR is

\[ R_t = S_t + Q_t - b \]

where $b_t$ is a deterministic parameter called the decision parameter.
In this LDR, the entire amount, $Q_t$, is taken into account while making the release decision. Depending on the proportion of inflow, $Q_t$, used in the linear decision rule, a number of such LDRs may be formulated. A general form of this LDR may be written as

$$R_t = S_t + \beta_t Q_t - b_t$$

$0 \leq \beta_t \leq 1$

$\beta_t = 0$ yields a relatively conservative release policy with release decisions related only to the storage, $S_t$; $\beta_t = 1$ yields an optimistic policy where the entire amount of water available ($S_t + Q_t$), is used in the LDR.

Consider the LDR

$$R_t = S_t + Q_t - b_t$$  \hspace{1cm} (6.2.4)

The storage continuity equation is

$$S_{t+1} = S_t + Q_t - R_t$$  \hspace{1cm} (6.2.5)

From (6.4) and (6.5)

$$S_{t+1} = b_t$$  \hspace{1cm} (6.2.6)

Thus, the random variable, $S_{t+1}$, is set equal to a deterministic parameter $b_t$. In essence, the role of the linear decision rule in this case is to treat $S_t$ deterministic in formulation. A main advantage of doing this is to do away with one of the random variables, $S_t$, so that the distribution of the other random variable, $R_t$, may be expressed in terms of the known distribution of $Q_t$. This implies that the variance of $Q_t$ is entirely transferred to the variance of $R_t$.

Including evaporation loss as a storage-dependent term in the storage continuity equation, the linear decision rule Eq. (6.2.4) is written as,

$$R_t = Q_t - [A_0 e_t^f] + [1 - (a e_t/2)] b_{t-1} - [1 + (a e_t/2)] b_t.$$  \hspace{1cm} (6.2.7)

where $A_0$, $a$, and $e_t$ are as defined earlier in Section 5.1.

### 6.2.2 Deterministic Equivalent of a Chance Constraint

Knowing the probability distribution of inflow, $Q_t$, it is possible to obtain the deterministic equivalents of the chance constraints using the LDR, (6.2.4), as follows:

\begin{align*}
\mathbb{P}[R_t \geq D_t] \geq \alpha_t \\
\mathbb{P}[S_t + Q_t - b_t \geq D_t] \geq \alpha_t \\
\mathbb{P}[S_{t+1} + Q_t - b_t \geq D_t] \geq \alpha_t \text{ using } S_t = b_t \text{ from (6.2.6)} \\
\mathbb{P}[Q_t \geq D_t + b_t - b_{t-1}] \geq \alpha_t \\
\mathbb{P}[Q_t \leq D_t + b_t - b_{t-1}] \leq 1 - \alpha_t \text{ see Fig. 6.9} \hspace{1cm} (6.2.7)
\end{align*}

**Fig. 6.9** Probability Density Function of $Q_t$
The term, $D_t + b_t - b_{t-1}$, on the right-hand side of the inequality within brackets in Eq. (6.2.7) is deterministic, with $b_{t-1}$ and $b_t$ being decision variables and the demand $D_t$ being a known quantity for the period $t$. Therefore Eq. (6.2.7) is written as

$$F_{Q_t}(D_t + b_t - b_{t-1}) \leq 1 - \alpha_t$$

where $F_{Q_t}(D_t + b_t - b_{t-1})$ denotes the CDF of the random variable $Q_t$. From this we write the deterministic equivalent of the chance constraint, $P[R_t \geq D_t] \geq \alpha_t$, as

$$(D_t + b_t - b_{t-1}) \leq F_{Q_t}^{-1}(1 - \alpha_t)$$

$F_{Q_t}^{-1}(1 - \alpha_t)$ is the flow, $q_t$, at which the CDF value is $1 - \alpha_t$, as shown in Fig. 6.10.

$$F_{Q_t}(1 - \alpha_t)$$

$$q_t$$

Fig. 6.10 Flow Value for a Specified CDF Value

The deterministic equivalent of the chance constraint, Eq. (6.2.1), is similarly obtained, as

$$R_t^{max} + b_t - b_{t-1} \geq F_{Q_t}^{-1}(\alpha_t)$$

Since the storage, $S_t$, is set equal to the deterministic parameter, $b_{t-1}$, the chance constraints (6.2.2) and (6.2.3), containing only the storage random variable are written as deterministic constraints (without using the probability distribution of inflows). The complete deterministic equivalent of the CCLP is thus written as

$$\text{Min } K$$

subject to

$$D_t + b_t - b_{t-1} \leq F_{Q_t}^{-1}(1 - \alpha_t) \quad \forall t$$

$$R_t^{max} + b_t - b_{t-1} \geq F_{Q_t}^{-1}(\alpha_t) \quad \forall t$$

$$b_{t-1} \leq K \quad \forall t$$

$$b_{t-1} \geq S_{min} \quad \forall t$$

$$b_t \geq 0 \quad \forall t$$

$$K \geq 0$$

While solving this model, for a problem with 12 periods (months) in a year, we also set $b_0 = b_{12}$ for a steady state solution. Further, depending on the nature of LDR used, the decision parameters, $b_t$, may be unrestricted in sign. For example, if we use the LDR, $R_t = S_t - b_t$, the decision parameter, $b_t$, may be
allowed to take negative values. When a particular $b_t$ is negative, it implies that the release is more than the initial storage in period $t$, and the volume of release over the available storage, $S_t$, is provided by part of the inflow, $Q_t$, not included in the LDR.

**Example 6.2.1** With the linear decision rule, $R_t = S_t + (1 - \alpha)Q_t - b_t$, where $0 \leq \alpha \leq 1$, obtain the deterministic equivalent of the storage chance constraint, $P(S_t \leq K) \geq 0.9$. Assume that the flows, $Q_t$, follow exponential distribution, with pdf given by, $f(q) = e^{-q/q}$, $q > 0$.

Write down the deterministic equivalent for a two-period ($t = 1$ and $t = 2$) problem when $\beta = 3$ for period $t = 1$, and $\beta = 5$ for period $t = 2$. $K$ is the known reservoir capacity. Neglect evaporation losses.

**Solution:**

LDR $R_t = S_t + (1 - \alpha)Q_t - b_t$

$LDR$ $S_{t+1} = S_t + Q_t - K$

$LDR$ $S_t = \alpha Q_t + b_t$

Deterministic equivalent of $P(S_t \leq K) \geq 0.9$

$P(\alpha Q_t + b_t \leq K) \geq 0.9$

$P(\alpha Q_t \leq K - b_t) \geq 0.9$

$P(Q_t \leq (K - b_t)/\alpha) \geq 0.9$

$(K - b_t)/\alpha \geq F_Q^{-1}(0.9)$

CDF of exponential distribution is $F(q) = 1 - e^{-q/q}$

For $t = 1$

$(K - b_2)/\alpha \geq F_Q^{-1}(0.9)$

From the distribution

$1 - e^{-q_2} = 0.9$

$q_2 = F_Q^{-1}(0.9) = 0.4605$

For $t = 2$

$(K - b_1)/\alpha \geq F_Q^{-1}(0.9)$

From the distribution

$1 - e^{-q_1} = 0.9$

$q_1 = F_Q^{-1}(0.9) = 0.7653$

Thus, the deterministic equivalents are

$(K - b_2)/\alpha \geq 0.4605 \quad \ldots t = 1$

$(K - b_1)/\alpha \geq 0.7653 \quad \ldots t = 2$
Example 6.2.2 Write down the complete deterministic equivalent of the following, three-period chance constrained LP problem. Use linear decision rule, \( R_t = S_t - b_t \).

\[
\begin{align*}
\text{Minimize} & \quad K \\
\text{subject to} & \quad P[S_{\text{min}} \leq S_t \leq K] \geq 0.9 \quad \forall t \\
& \quad P[R_t \leq R_{\text{max}}] \geq 0.95 \quad \forall t \\
& \quad P[R_t \geq 0] \geq 0.75 \quad \forall t \\
\end{align*}
\]

\( S_t \) is the storage at the beginning of the period \( t \), \( R_t \) is the release during the period \( t \), and \( K \) is the reservoir capacity. Neglect losses. The following table gives the \( F^{-1}(\cdot) \) values for the inflows and the \( R_{\text{max}} \) and \( R_{\text{min}} \) values for different periods. Solve the CCLP problem to obtain the minimum capacity required.

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Solution:

LDR is \( R_t = S_t - b_t \)

\[
\begin{align*}
S_{t+1} &= S_t + Q_t - R_t \\
&= S_t + Q_t - S_t + b_t \\
&= Q_t + b_t \\
S_t &= Q_{t+1} + b_t \\
\end{align*}
\]

Deterministic equivalent of

\[
\begin{align*}
P[S_{\text{min}} \leq S_t \leq K] & \geq 0.9 \\
P[S_{\text{min}} \leq S_{t+1}] & \geq 0.9 \\
P[S_t] & \geq 0.9 \\
\end{align*}
\]

Deterministic equivalent of

\[
\begin{align*}
P[S_{\text{min}} \leq S_t] & \geq 0.9 \\
P[S_{\text{min}} \leq Q_{t+1} + b_t] & \geq 0.9 \quad \text{(as } S_{\text{min}} = 2\text{)} \\
P[Q_{t+1} \geq S_{\text{min}} - b_t] & \geq 0.9 \\
P[Q_{t+1} \leq S_{\text{min}} - b_t] & \leq (1 - 0.9) \\
P[Q_{t+1} \leq 2 - b_t] & \leq 0.1 \\
F_{Q_{t+1}}(2 - b_t) & \leq 0.1 \\
2 - b_t & \leq F_{Q_{t+1}}^{-1}(0.1) \\
2 - b_t & \leq F_{Q_{t+1}}^{-1}(0.1) \quad \text{... for } t = 1 \\
2 - b_t & \leq F_{Q_{t+1}}^{-1}(0.1) \quad \text{... for } t = 2 \\
2 - b_t & \leq F_{Q_{t+1}}^{-1}(0.1) \quad \text{... for } t = 3
\end{align*}
\]
From the $F^{-1}()$ values given, 
\[ 2 - b_t \leq 6 \quad t = 1 \]
\[ 2 - b_t \leq 12 \quad t = 2 \]
\[ 2 - b_t \leq 3 \quad t = 3 \]

Deterministic equivalent of 
\[ P[S_i \leq K] \geq 0.9 \]
\[ P[Q_{i-1} + b_{i-1} \leq K] \geq 0.9 \]
\[ P[Q_{i-1} \leq K - b_{i-1}] \geq 0.9 \]
\[ F_{Q_{i-1}'}(2 - b_{i-1}) \leq 0.1 \]
\[ K - b_{i-1} \geq F_{Q_{i-1}'}^{-1}(0.9) \]
Thus,
\[ K - b_{2} \geq 72 \quad \ldots \quad t = 1 \]
\[ K - b_{1} \geq 90 \quad \ldots \quad t = 2 \]
\[ K - b_{2} \geq 60 \quad \ldots \quad t = 3 \]

Deterministic equivalent of 
\[ P[R_t \leq R_{\text{max}}] \geq 0.95 \]
\[ P[S_t - b_t \leq R_{\text{max}}] \geq 0.95 \]
\[ P[Q_{i-1} + b_{i-1} - b_t \leq R_{\text{max}}] \geq 0.95 \]
\[ P[Q_{i-1} \leq R_{\text{max}} + b_t - b_{i-1}] \geq 0.95 \]
\[ F_{Q_{i-1}'}(R_{\text{max}} + b_t - b_{i-1}) \geq 0.95 \]
\[ R_{\text{max}} + b_t - b_{i-1} \geq F_{Q_{i-1}'}^{-1}(0.95) \]
\[ 90 + b_{1} - b_{1} \geq F_{Q_{i-1}'}^{-1}(0.95) \quad \ldots \quad t = 1 \]
\[ 84 + b_{2} - b_{1} \geq F_{Q_{i-1}'}^{-1}(0.95) \quad \ldots \quad t = 2 \]
\[ 84 + b_{3} - b_{2} \geq F_{Q_{i-1}'}^{-1}(0.95) \quad \ldots \quad t = 3 \]

With the values given,
\[ 90 + b_{1} - b_{1} \geq 85 \quad \ldots \quad t = 1 \]
\[ 84 + b_{2} - b_{1} \geq 93 \quad \ldots \quad t = 2 \]
\[ 84 + b_{3} - b_{2} \geq 80 \quad \ldots \quad t = 3 \]

Deterministic equivalent of 
\[ P[R_t \leq D] \geq 0.75 \]
\[ P[Q_{i-1} + b_{i-1} - b_t \leq D_t] \geq 0.75 \]
\[ P[Q_{i-1} \leq D_t + b_t - b_{i-1}] \geq 0.75 \]
\[ F_{Q_{i-1}'}(D_t + b_t - b_{i-1}) \leq (1 - 0.75) \]
\[ F_{Q_{i-1}'}^{-1}(D_t + b_t - b_{i-1}) \leq 0.25 \]
\[ D_t + b_t - b_{i-1} \leq F_{Q_{i-1}'}^{-1}(0.25) \]
With the values given,
\[
\begin{align*}
24 + b_1 - b_3 &\leq F_{Q_1}^{-1}(0.25) & t = 1 \\
20 + b_2 - b_3 &\leq F_{Q_2}^{-1}(0.25) & t = 2 \\
20 + b_3 - b_2 &\leq F_{Q_3}^{-1}(0.25) & t = 3 \\
24 + b_1 - b_3 &\leq 21 & t = 1 \\
20 + b_2 - b_3 &\leq 33 & t = 2 \\
20 + b_3 - b_2 &\leq 20 & t = 3
\end{align*}
\]

The solution of this model results in
\[
K = 90; \quad b_1 = 0; \quad b_2 = 9; \quad b_3 = 5.
\]

**Example 6.2.3** Formulate a CCLP model to obtain minimum capacities of the reservoirs in the four-reservoir system shown in Fig. 6.11 and write down the deterministic equivalents, assuming that all water released from an upstream reservoir is available at the downstream reservoir. A minimum flood free-board is to be maintained in all reservoirs in each period.

Fig. 6.11 | A four-reservoir System

Capacities of the reservoirs are \(K_1, K_2, K_3,\) and \(K_4\) respectively. We use the following notation in this example:

- \(S_i\) : Storage in reservoir \(i\) at the beginning of period \(t\).
- \(R_i\) : Release from reservoir \(i\) in period \(t\).
- \(Q_i\) : Inflow to reservoir \(i\) in period \(t\).
- \(b_i\) : Decision parameter for reservoir \(i\), period \(t\).
Model Development

\( y_i \): Minimum flood storage to be maintained in reservoir \( i \), at the beginning of period \( t \).

\( R_{i \text{ min}} \): Minimum release from reservoir \( i \), in period \( t \).

\( R_{i \text{ max}} \): Maximum release from reservoir \( i \), in period \( t \).

\( S_{i \text{ min}} \): Storage below which no release is made from reservoir \( i \), in period \( t \).

Linear decision rule (LDR):

Reservoirs 1 and 3

\[ R_i = S_i - b_i \quad i = 1, 3 \]

Reservoir 2

\[ R_2 = S_2 + R_{i1} - b_2 \quad (R_{i1} \text{ is upstream release}) \]

Reservoir 4

\[ R_4 = S_4 + R_{i2} + R_{i3} - b_4 \]

Substituting these in storage continuity equation

Reservoirs 1 and 3

\[ S_{i1} = S_i + Q_i - R_i \]

\[ = S_i + Q_i - S_i + b_i \]

\[ = Q_i + b_i \quad i = 1, 3 \]

\[ S_i = Q_{i, i1} + b_{i, i1} \]

\[ R_i = Q_{i, i2} + b_{i, i2} - b_i \]

Reservoir 2

\[ S_{2, i} = S_2 + Q_2 + R_{i1} - R_2 \]

\[ = S_2 + Q_2 + R_{i1} - S_2 + R_{i1} + b_2 \]

\[ = Q_2 + b_2 \]

\[ S_2 = Q_{2, i2} + b_{2, i2} \]

\[ R_2 = S_2 + R_{i2} - b_2 \]

\[ = \sum_{i=1}^{2} Q_{i, i1} + \sum_{i=1}^{2} [b_{i, i1} - b_i] \]

substituting for \( R_{i1} \)

Reservoir 4

\[ S_4 = Q_{4, i4} + b_{4, i4} \]

\[ R_4 = S_4 + R_{i3} + R_{i4} - b_4 \]

\[ = \sum_{i=1}^{2} Q_{i, i4} + \sum_{i=1}^{2} [b_{i, i4} - b_i] \]

substituting for \( R_{i3} \) and \( R_{i4} \)

\( S_i \) is the initial storage for the reservoir \( i \) at period \( t \). \( y_i \) is the freeboard to be maintained at the end of period \( t \).

The CCLP is written as

\[ \text{Min } K_1 + K_2 + K_3 + K_4 \]

\[ P[S_i \geq S_{i \text{ min}}] \geq \alpha_{i, i} \quad \forall i, t \]

\[ P[K_i - S_i \geq y_i] \geq \alpha_{2, i} \quad \forall i, t \]
Reservoir Systems—Random Inflow

\[ P[R_i \leq R'_{i \text{max}}] \geq \alpha_{i,t}^{'} \quad \forall i, t \]

\[ P[R_i \geq R'_{i \text{max}}] \geq \alpha_{i,t}^{'} \quad \forall i, t \]

**Deterministic Equivalent**

Min \( K_1 + K_2 + K_3 + K_4 \)

subject to

**Reservoir 1**

\[ s_{i \text{min}} b_{i-1} \leq F_{Q_i-1}^{-1} (1 - \alpha_{i,1}^{''}) \]

\[ K_1 = b_{i-1} - y_i \geq F_{Q_i-1}^{-1} (1 - \alpha_{i,2}^{''}) \]

\[ k_{i \text{min}} b_{i-1} + b_t \leq F_{Q_i-1}^{-1} (1 - \alpha_{i,3}^{''}) \]

\[ k_{i \text{max}} b_{i-1} + b_t \geq F_{Q_i-1}^{-1} (\alpha_{i,3}^{''}) \]

**Reservoir 2**

\[ s_{i \text{min}} b_{i-1} \leq F_{Q_i-1}^{-1} (1 - \alpha_{i,1}^{''}) \]

\[ K_2 = b_{i-1} - y_i \geq F_{Q_i-1}^{-1} (\alpha_{i,2}^{''}) \]

\[ k_{i \text{min}} b_{i-1} + b_t \leq F_{Q_i-1}^{-1} (1 - \alpha_{i,3}^{''}) \]

\[ k_{i \text{max}} b_{i-1} + b_t \geq F_{Q_i-1}^{-1} (\alpha_{i,3}^{''}) \]

Note that \( F_{Q_i-1}^{-1} (\cdot) \) is obtained from the distribution of the sum of the flows, viz. \( Q_{i-1} + Q_{i-1} \).

**Reservoir 3**

\[ s_{i \text{min}} b_{i-1} \leq F_{Q_i-1}^{-1} (1 - \alpha_{i,1}^{''}) \]

\[ K_3 = b_{i-1} - y_i \geq F_{Q_i-1}^{-1} (\alpha_{i,2}^{''}) \]

\[ k_{i \text{min}} b_{i-1} + b_t \leq F_{Q_i-1}^{-1} (1 - \alpha_{i,3}^{''}) \]

\[ k_{i \text{max}} b_{i-1} + b_t \geq F_{Q_i-1}^{-1} (\alpha_{i,3}^{''}) \]

**Reservoir 4**

\[ s_{i \text{min}} b_{i-1} \leq F_{Q_i-1}^{-1} (1 - \alpha_{i,1}^{''}) \]

\[ K_4 = b_{i-1} - y_i \geq F_{Q_i-1}^{-1} (\alpha_{i,2}^{''}) \]

\[ k_{i \text{min}} b_{i-1} + b_t \leq F_{Q_i-1}^{-1} (1 - \alpha_{i,3}^{''}) \]

\[ k_{i \text{max}} b_{i-1} + b_t \geq F_{Q_i-1}^{-1} (\alpha_{i,3}^{''}) \]

\[ \sum_{i=1}^{4} (1 - \alpha_{i,4}^{''}) \]
Model Development

\[ K_{\text{max}} - b_{3t-1} - b_{2t-1} - b_{1t-1} + b_{3t} + b_{2t} + b_{1t} \]
\[ \geq F^{-1} \left( \sum_{t=1}^{\alpha(t)} b_{t} \right) \]

Note that \( F^{-1}(\cdot) \) appearing on the RHS of the last 2 constraints of Reservoir 4 is obtained from the distribution of the sum of the flows, \( Q_{1t-1} + Q_{2t-1} + Q_{3t-1} + Q_{4t-1} \).

**Example 6.2.4** Relevant data for the multireservoir problem discussed in Example 6.2.3 are given in the following tables. The configuration of the system and the data are adapted from Nayak and Arora (1971), with minor modifications. All \( \alpha(t) \) values are set to 0.90. Flows corresponding to \( Fq_{1}(1 - \alpha) \) are denoted as \( Fq_{\text{min}} \) and those corresponding to \( Fq_{1}(\alpha) \) as \( Fq_{\text{max}} \) for convenience. Similarly \( Fq_{2\text{min}}, Fq_{2\text{max}} \), \( Fq_{3\text{min}}, Fq_{3\text{max}} \), and \( Fq_{4\text{min}}, Fq_{4\text{max}} \) are denoted as \( Fq_{2\text{min}}, Fq_{2\text{max}} \), \( Fq_{3\text{min}}, Fq_{3\text{max}} \), and \( Fq_{4\text{min}}, Fq_{4\text{max}} \).

(Units for flows, releases, decision parameters, and storages: Mm³)

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Reservoir 3

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</table>

This problem is solved using LINGO (PC software for LP). The solution obtained is given in the following table.

Solution of the Multireservoir CCLP Problem (All Values in Mm$^3$)

<table>
<thead>
<tr>
<th>Reservoir 1</th>
<th>Reservoir 2</th>
<th>Reservoir 3</th>
<th>Reservoir 4</th>
<th>Total Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>optimal capacity month</td>
<td>366.85</td>
<td>656.65</td>
<td>691.9</td>
<td>570.75</td>
</tr>
</tbody>
</table>

Decision Parameters, $b_i$

<table>
<thead>
<tr>
<th>Monthly</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
<th>$b_4$</th>
<th>$b_5$</th>
<th>$b_6$</th>
<th>$b_7$</th>
<th>$b_8$</th>
<th>$b_9$</th>
<th>$b_{10}$</th>
<th>$b_{11}$</th>
<th>$b_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.75</td>
<td>-3.75</td>
<td>-1.15</td>
<td>-1.15</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>-1.15</td>
<td>-7.85</td>
<td>-6.4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>-9.95</td>
<td>-13.15</td>
<td>-9.8</td>
<td>-24.95</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0.85</td>
<td>-243.55</td>
<td>25</td>
<td>-4.35</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>140.85</td>
<td>-59.35</td>
<td>119.9</td>
<td>227.75</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>153.85</td>
<td>13.65</td>
<td>148.9</td>
<td>247.75</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>149.85</td>
<td>25.55</td>
<td>171.9</td>
<td>232.85</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>144.45</td>
<td>64.95</td>
<td>190.2</td>
<td>36.55</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>138.15</td>
<td>64.25</td>
<td>107</td>
<td>-11.25</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>84.95</td>
<td>-18.85</td>
<td>94.3</td>
<td>-11.25</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>61.55</td>
<td>-11.45</td>
<td>74.5</td>
<td>-7.45</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>12</td>
<td>15.65</td>
<td>18.35</td>
<td>55.6</td>
<td>-6.45</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Problems

6.2.1 At a site proposed for a reservoir, the streamflows for the two seasons within a year are known to be normally distributed with the following parameters:

<table>
<thead>
<tr>
<th>Season</th>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>30</td>
<td>20</td>
</tr>
<tr>
<td>Std. Dev</td>
<td>18</td>
<td>12</td>
</tr>
</tbody>
</table>

The reservoir is required to serve demands of 20 and 30 units in the two seasons, respectively. Formulate a chance constrained linear programming problem to determine the minimum capacity of the reservoir. Using an LDR of the form, $S_t = R_t - b_t$, with the usual notations, write down the deterministic equivalent of the minimum release constraints for the two seasons. The minimum reliability with which the demands must be met in the two periods are 0.95 and 0.9, respectively.

6.2.2 Ten years of data are available for the inflows at a site, for the two seasons ($t = 1$ and $t = 2$) in a year. The inflow values (Mm$^3$) are given below:

- $t = 1$: 1500, 4410, 1020, 8170, 11750, 5609, 6579, 1098, 3200, 3430
- $t = 2$: 1000, 787, 1020, 3040, 779, 678, 908, 2200, 1690, 1219

Average evaporation during the two seasons is 3.0 and 1.0 Mm$^3$ respectively. It is known that the inflows in season $t = 1$ are lognormally distributed, and those in season $t = 2$ are normally distributed. With the usual notations used in Chance Constrained LP for reservoir design, write down the complete deterministic equivalents of the following chance constraints. Use the LDR, $S_t = R_t + b_t$.

- $P[R_t \geq 3000] \geq 0.85 \quad t = 1, 2$
- $P[R_t \leq 7000] \geq 0.95 \quad t = 1, 2$

6.2.3 Monthly flows at a site proposed for a reservoir are normally distributed with the following parameters:

<table>
<thead>
<tr>
<th>Jun</th>
<th>Jul</th>
<th>Aug</th>
<th>Sep</th>
<th>Oct</th>
<th>Nov</th>
<th>Dec</th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
<th>Apr</th>
<th>May</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>624</td>
<td>3311</td>
<td>3638</td>
<td>1534</td>
<td>1196</td>
<td>495</td>
<td>207</td>
<td>98</td>
<td>55</td>
<td>39</td>
<td>52</td>
</tr>
<tr>
<td>Std.</td>
<td>468</td>
<td>1889</td>
<td>1391</td>
<td>629</td>
<td>512</td>
<td>273</td>
<td>106</td>
<td>86</td>
<td>33</td>
<td>38</td>
<td>50</td>
</tr>
</tbody>
</table>

Formulate and solve a chance constrained LP problem to determine the minimum reservoir capacity required to satisfy a constant demand of 0.95A, A, and 1.05A (where A is the average of the mean flows). Solve the problem for a reliability of 90% of meeting the demand. Use a convenient LDR.

6.3 CONCEPT OF RELIABILITY

Reliability is essentially a measure of chance. When we talk about reliability, we mean a measure of an outcome from an operation. In irrigation, for example, we may think of reliability as a measure of assurance with which water is supplied to satisfy the crop water requirement. In flood control, it could be
the level of assurance that the submergence in the affected area is limited to be within a specified level. A criterion for reliability and an example illustrating its application are presented in this section.

Reliability Criterion In this section, reliability with reference to reservoir operation for irrigation is discussed. In reservoir operation for irrigation, it is important to determine the maximum level of reliability that an existing system can be operated at, and also to find the minimum reservoir capacity required for a desired or specified level of reliability. The reliability, in this case, could be expressed as a measure of the probability that the quantum of water supplied from the reservoir release is equal to or greater than the total irrigation demand. We shall also specify that this reliability should be a minimum of \( \alpha \). This is done through a chance constraint

\[
\Pr \left[ R_t \geq \frac{1}{\eta} \sum c_t A'(c) \right] \geq \alpha
\]

where,
- \( R_t \) = release from the reservoir in period, \( t \)
- \( \eta \) = a parameter equal to the ratio of the total crop water allocation to the reservoir release at the plant level (irrigation efficiency)
- \( c_t \) = irrigation allocation to crop \( c \) in period \( t \), in depth units
- \( A' \) = area of crop \( c \), in area units
- \( \alpha \) = reliability (specified minimum)

The chance constraint may be used as a reliability constraint in application to determine:
1. the minimum size of the reservoir to meet irrigation requirements at a specified level of reliability, and
2. the maximum possible reliability of meeting crop irrigation requirements for a given reservoir capacity (as in an existing reservoir).

The chance constraint is expressed in terms of its deterministic equivalent in a chance constrained linear program (CCLP). The reservoir release, \( R_c \), can be related to the random inflow through the reservoir storage continuity equation and a linear decision rule for the period \( t \) (see Section 6.2.1).

6.3.1 Reliability-based Reservoir Sizing

Consider a case where the total irrigation demand, \( D_t \), at the reservoir in each period, \( t \), is known, as estimated from relevant field data. Then the reliability constraint may be written as

\[
\Pr [R_t \geq D_t] \geq \alpha_{t}
\]

where \( R_t \) is the reservoir release (for irrigation) and \( \alpha_{t} \) is the reliability level in period \( t \).

Following the notation of earlier sections, the storage continuity equation is

\[
R_t = (1 - a_t) S_t + Q_t - L_t - (1 + a_t) S_{t+1}
\]

and from the linear decision rule

\[
S_{t+1} = b_t, \text{ where } b_t \text{ is a nonrandom parameter.}
\]
Thus,
\[ R_t = (1 - a_t)b_t - 1 + Q_t - L_t. \]

In this equation, \( R_t \) and \( Q_t \) are the only random variables, as the linear decision rule transformed the random storage into a deterministic variable, absorbing all the randomness of inflow into the release term, \( R_t \).

The chance constraint now becomes
\[ \text{Prob}[Q_t - L_t - (1 + e_t)b_t] \geq \alpha_t. \]

The deterministic equivalent of this chance constraint is
\[ (1 + e_t)b_t - (1 - a_t)b_t - 1 + L_t + D_t \leq F^{-1}(1 - \alpha_t). \]

where \( F^{-1}(1 - \alpha_t) = Q_t^{\text{max}} \) = inflow in period \( t \), with a probability of \( (1 - \alpha_t) \), or an exceedance probability of \( \alpha_t \).

The storage, \( S_t = b_t - 1 \), is limited to the active storage capacity of the reservoir in each period, \( b_t - 1 \), and \( b_0 = b_1 \) for a steady state solution.

The model for optimizing the reservoir capacity, thus, is
\[
\begin{align*}
\text{minimize} & \quad K \\
\text{subject to} & \quad (1 + a_t)b_t - (1 - a_t)b_t - 1 + L_t + D_t \leq Q_t^{\text{max}} - 1 \\
& \quad b_{t+1} \leq K, \quad t = 1, 2, \ldots, T, \\
& \quad b_0 = b_1. 
\end{align*}
\]

The model will be simplified if the reliability level is the same for all periods, \( \alpha_t = \alpha \) for all \( t \). The model is solved using linear programming for a specified value of \( \alpha \). The solution gives the minimum required capacity of the reservoir corresponding to the specified value of \( \alpha \).

**Maximum Reliability**

A relationship between \( \alpha \) and the corresponding \( K \) may be plotted for different values of reliability to analyze the implied trade-off. It may be noted that beyond some value of \( \alpha \), the solution will be infeasible, as the reliability of irrigation supply for the given inflow data cannot be indefinitely increased by merely increasing the reservoir capacity. This is a condition arising from limited inflows. If the model is solved for different discrete but increasing values of \( \alpha \) (each solution corresponding to one value of \( \alpha \)), there will be a stage when beyond, say, \( \alpha = \alpha_{\text{max}} \), the solution becomes infeasible. It means that \( \alpha_{\text{max}} \) then is the maximum feasible reliability for the given inflow distribution. Let \( K^* \) be the corresponding capacity requirement (corresponding to \( \alpha = \alpha_{\text{max}} \)). While the maximum feasible reliability is determined by the inflow data, the maximum attainable reliability, however, will be limited due to either limited inflows, or limited capacity in the case of a reservoir of specified capacity.

If the analysis is performed for an existing reservoir of capacity, \( K_o \), one can find if the reservoir is oversized (\( K_o > K^* \)) or undersized (\( K_o < K^* \)) with reference to realizing the possible maximum reliability of meeting the demand. In the former case (reservoir oversized), it is inflow that limits the maximum reliability, and in the latter case (reservoir undersized) it is the
capacity. In the former case, the capacity in excess of \( K^* \) is not useful, as it does not contribute to any increase in the reliability over and above \( \alpha_{\text{max}} \) (restricted by the inflow limitation). In the latter case \( (K_0 < K^*) \), however, the existing capacity is not enough to realize the maximum possible reliability. This is a condition of capacity limitation. In this case \( (K_0 < K^*) \), it is possible to improve the reliability beyond \( \alpha_0 \) (corresponding to \( K_0 \)), by providing additional capacity over and above the existing capacity, \( K_0 \).

Thus a plot between \( K \) and \( \alpha \) gives an insight into the performance of a reservoir in terms of whether it is inflow or capacity that might limit the maximum attainable reliability level for the given inflow data.

**Example 6.3.1** For an existing reservoir of capacity 1320 Mm\(^3\), with given monthly inflow and demand values for each month, it is determined, by running CCLP, that a capacity of 1253 Mm\(^3\) is required to get the maximum possible reliability of 0.825.

Also, capacities required for different specified reliabilities are obtained, as shown in the following table, by running CCLP with different values of reliability and the corresponding sequence of monthly inflows, keeping the demands the same.

<table>
<thead>
<tr>
<th>Reliability ( \alpha )</th>
<th>Required capacity ( K ) (Mm(^3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>1056</td>
</tr>
<tr>
<td>0.7</td>
<td>1159</td>
</tr>
<tr>
<td>0.8</td>
<td>1238</td>
</tr>
<tr>
<td>0.825</td>
<td>1253</td>
</tr>
</tbody>
</table>

(i) Plot the capacity–reliability relationship.
(ii) Interpret the results of these runs in terms of the inflow or capacity limitation that restricts the maximum attainable reliability of meeting the demands.

**Solution:**
(i) the \( K-\alpha \) plot is as shown in the following figure.

(ii) From the data given, \( \alpha_{\text{max}} = 0.825 \), which means that runs made for values higher than this are found to be infeasible. The capacity

corresponding to this, $K^* = 1253 \text{ Mm}^3$, is less than the existing capacity, $K_o = 1320 \text{ Mm}^3$. Therefore, in this case, the inflows limit the maximum attainable reliability. This is a case of inflow limitation.

Instead, if the actual capacity were to be 1159 \text{ Mm}^3, the maximum attainable reliability would have been limited to only 0.7, and the case would have been one of capacity limitation.

### Problems

6.3.1 The mean and standard deviations of monthly inflows to a reservoir are given below.

<table>
<thead>
<tr>
<th>Month</th>
<th>Mean (Mm$^3$)</th>
<th>Std. dev. (Mm$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jun</td>
<td>267.7</td>
<td>165.5</td>
</tr>
<tr>
<td>Jul</td>
<td>960.1</td>
<td>355.7</td>
</tr>
<tr>
<td>Aug</td>
<td>835.6</td>
<td>337.8</td>
</tr>
<tr>
<td>Sep</td>
<td>359.9</td>
<td>161.1</td>
</tr>
<tr>
<td>Oct</td>
<td>264.5</td>
<td>122.9</td>
</tr>
<tr>
<td>Nov</td>
<td>128.8</td>
<td>72.5</td>
</tr>
<tr>
<td>Dec</td>
<td>73.50</td>
<td>53.0</td>
</tr>
<tr>
<td>Jan</td>
<td>35.40</td>
<td>15.5</td>
</tr>
<tr>
<td>Feb</td>
<td>18.80</td>
<td>7.2</td>
</tr>
<tr>
<td>Mar</td>
<td>13.10</td>
<td>5.6</td>
</tr>
<tr>
<td>Apr</td>
<td>13.80</td>
<td>5.8</td>
</tr>
<tr>
<td>May</td>
<td>26.50</td>
<td>26.1</td>
</tr>
</tbody>
</table>

Assume that monthly flows are normally distributed.

Prepare sets of monthly flows such that, in a given set, the inflow in each month has an exceedance probability, $\alpha$ (or cumulative probability, $1 - \alpha$). Prepare these sets of inflows, corresponding to $\alpha = 0.6, 0.7, \text{ and } 0.8$, respectively.

6.3.2 Obtain data of the mean and the standard deviation of monthly inflows to a nearby reservoir. Assume that the inflow in each month is normally distributed. Generate inflows in each month for different levels of exceedance probability. Get also monthly demand and evaporation data.

1. Formulate a CCLP model and determine the minimum required capacity, $K$, for different levels of $\alpha$.
2. Determine $\alpha_{\text{max}}$ and the corresponding $K^*$.
3. Plot the relationship between $K$ and $\alpha$.
4. Determine, for the particular reservoir studied, whether the attainable reliability is conditioned by the inflow limitation or the capacity limitation.

6.3.3 Obtain the following data for an existing reservoir.

- Monthly inflow data for the reservoir, $Q_t$, for a few years, say, about 20 years.
- Monthly water demands (to be released from the reservoir), $D_t$.
- Existing reservoir capacity, $K_o$.
6.4 Stochastic Dynamic Programming for Reservoir Operation

Stochastic Dynamic Programming (SDP) belongs to the Explicit Stochastic Optimization (ESO) class of optimization models. In this section, we discuss SDP with specific application to the reservoir operation problem, in which the inflow to the reservoir is considered as a random variable.

6.4.1 State Variable Discretization

The reservoir operation problem is a complex real world control problem in which decisions need to be taken sequentially in time, based on the known state of the system. In the following sections, we shall develop a discrete variable SDP model. All variables involved in the decision process, such as the reservoir storage, inflow, and release are discretized into a finite number of class intervals. A class interval for a variable has a representative value, generally taken as its midpoint. The reservoir storage at the beginning of period $t$ and inflow during the period $t$ are treated as state variables.

The following notation is followed in the SDP formulation: $Q$ denotes the inflow; $i$ and $j$ are the class intervals (also referred to as states) of inflow in period $t$ and period $t + 1$, respectively; $S$ denotes the reservoir storage; and $k$ and $l$ are the storage class intervals in periods $t$ and $t + 1$, respectively. The representative values of inflow for the class $i$ in period $t$ and class $j$ in period $t + 1$ are denoted by $Q_{it}$ and $Q_{jt+1}$, respectively. Similarly, the representative values for storage in the class intervals $k$ and $l$ are denoted by $S_{kt}$ and $S_{lt+1}$, respectively. From the storage continuity, then, we may write

$$R_{kt} = S_{kt} + Q_{it} - E_{kt} - S_{lt+1}$$

(6.4.1)
where \( R_k \) is the reservoir release corresponding to the initial reservoir storage \( S_{kt} \), the final reservoir storage \( S_{lt+1} \), and the evaporation loss \( E_{kt} \). The loss \( E_{kt} \) depends on the initial and final reservoir storages, \( S_{kt} \) and \( S_{lt+1} \). Since the inflow \( Q \) is a random variable, the reservoir storage and the release are also random variables.

### 6.4.2 Inflow as a Stochastic Process

A stochastic process, denoted as \( \{X(t), t \in T\} \), is a collection of random variables, with \( t \) referring to time, \( X(t) \) the random variable, and \( T \) an index set. In the development of the SDP recursive equations, the reservoir inflow is treated as a stochastic process. Further, it is assumed that the reservoir inflows follow a first order Markov Chain. A stochastic process, \( \{X_n\} \), is said to be a first order Markov Chain if the dependence of future values of the process on the past values is completely determined by its dependence on the current value alone. A first order Markov Chain has the property,

\[
P[X_{t+1} = j | X_t = i, X_{t-1}, \ldots, X_0] = P[X_{t+1} = j | X_t = i]
\]

where the left-hand side (LHS) gives the conditional probability of \( X_{t+1} \) given the current and past values \( X_t, X_{t-1}, \ldots, X_0 \), and the right-hand side (RHS) gives the conditional probability of \( X_{t+1} \) given the current value \( X_t \) alone.

The assumption of a Markov Chain implies that the dependence of the inflow in the next period on the inflow during the current and all previous periods is completely described by its dependence on the inflow during the current period alone. Further, transition probabilities are used to measure the dependence of the inflow during period \( t + 1 \) on the inflow during period \( t \). With the notation defined above, the transition probability \( P_{ij}^t \) is defined as the probability that the inflow during the period \( t+1 \) will be in the class interval \( j \), given that the inflow during the period \( t \) lies in the class interval \( i \). That is,

\[
P_{ij}^t = P[Q_{t+1} = j | Q_t = i]
\]

where \( Q_t = i \) indicates that the inflow during the period \( t \) belongs to the discrete class interval \( i \). In applications, the transition probabilities, \( P_{ij}^t \), are estimated from historical inflow data. A suitable inflow discretization scheme is arrived at first. Each inflow value in the historical data set is then assigned the class interval to which it belongs. The number of times the inflow in period \( t+1 \) goes to class \( j \), when the inflow in the preceding period \( t \) belongs to class \( i \), divided by the number of times the inflow belongs to class \( i \) in period \( t \) is taken as the estimate of \( P_{ij}^t \). Note that for this relative frequency approach of estimating the transition probabilities, inflow data must be available for a sufficiently long length of time.

### 6.4.3 Steady State Operating Policy

The system performance measure depends on the state of the system defined by the storage class intervals \( K \) and \( L \), and the inflow class interval \( I \) for the period \( t \). We denote the system performance measure for period \( t \) as \( B_{ktl} \), which
corresponds to an initial storage state $k$, inflow state $i$, and final storage state $l$ in period $t$. The system performance measure may be, for example, the amount of hydropower generated when a release of $R_{klt}$ is made from the reservoir, and the reservoir storages (which determine the head available for power generation) at the beginning and end of the period are respectively $S_{kt}$ and $S_{lt+1}$. Following backward recursion, the computations are assumed to start at the last period $T$ of a distant year in the future and proceed backwards. Each time period denotes a stage in the dynamic programming. That is, $n = 1$ when $t = T$; $n = 2$ when $t = T - 1$, etc. The index $t$ takes values from $T$ to 1, and the index $n$ progressively increases with the stages in the SDP (Fig. 6.12). Let $f^T_t(k, i)$ denote the maximum expected value of the system performance measure up to the end of the last period $T$ (i.e. for periods $t, t+1, \ldots, T$), when $n$ stages are remaining, and the time period corresponds to $t$.

With only one stage remaining (i.e. $n = 1$ and $t = T$), we write,

$$ f^T_t(k, i) = \text{Max} \{ B_{kilt} \} \forall k, i \quad (6.4.3) $$

Note that for a given $k$ and $i$, only those values of $l$ are feasible that result in a non-negative value of release, $R_{kilt}$ (see Eq. 6.4.1). Since this is the last period in computation, the performance measure $B_{kilt}$ is determined with certainty for the known values of $k, i$ and $l$. When we move to the next stage, $(n = 2, t = T - 1)$, the maximum value of the expected performance of the system is written as

$$ f^{T-1}_t(k, i) = \text{Max} \{ B_{kilt} + \sum_j p_{ij} f^{T-1}_t(l, j) \} \forall k, i \quad (6.4.4) $$

When the computations are carried out for stage 2, period $T - 1$, the inflow $i$ during the period is known. However, since we are interested in obtaining the maximum expected system performance up to the end of the last period $T$, we must know the inflow during the succeeding period $T$ also. Since this is not known with certainty, the expected value of the system performance is got by using the inflow transition probabilities $p_{ij}^{T-1}$ for the period $T - 1$. It must be noted that the term within the summation in (6.4.4) denotes the maximized expected value of the system performance up to the end of the last period $T$, when the inflow state during the period $T - 1$ is $i$. The search for the optimum value of the performance is made over the end-of-the period storage $l$. Since $f^T_t(k, i)$ is already determined in stage 1, for all values of $k$ and $i$, $f^{T-1}_t(k, i)$ given by (6.4.4) may be determined. The term \{feasible $l$\}, appearing in (6.4.3)
and (6.4.4), indicates that the search is made only over those end-of-the-period storages which result in a non-negative release $R_k$, defined by (6.4.1).

The relationship (6.4.4) may be generalised for any stage $n$ and period $t$ as

$$P_k^t (k, i) = \text{Max} \{ R_{ij} + \sum_{l} f_{kl}^i (l, j) \} \quad \forall \; k, i \quad (6.4.5)$$

Solving (6.4.5) recursively will yield a steady state policy within a few annual cycles, if the inflow transition probabilities $P_{ij}$ are assumed to remain the same every year, which implies that the reservoir inflows constitute a stationary stochastic process. In general, the steady state is reached when the expected annual system performance, $\{ f_{kl}^i (k, i) - f_{kl}^i (k, j) \}$ remains constant for all values of $k$, $i$, and $t$. When the steady state is reached, the optimal end-of-the-period storage class intervals, $l$, are defined for given $k$ and $i$ for every period $t$ in the year. This defines the optimal steady state policy and is denoted by $l^* (k, i, t)$.

**Steady State Probabilities**

Assuming that we have a unique $l^* (k, i, t)$ for any given $k$, $i$ in period $t$, it is possible to obtain the steady state probabilities of the release as $PR_{kij}$, without the index $l$. For the two-state, single-step Markov chain considered, these probabilities are given by

$$PR_{kij} = \sum_{l} PR_{lij} P_{ij}^l \quad \forall \; l, j, t \quad (6.4.6)$$

$$l = l^*(k, i, t)$$

$$\sum_{l} PR_{lij} = 1 \quad \forall \; t \quad (6.4.7)$$

The RHS of (6.4.6) is a selective summation over only those initial storage and inflow indices $k$ and $i$ in period $t$ that result in the same $l = l^*(k, i, t)$. One equation in the set of equations (6.4.6) is redundant (in the light of Eq. 6.4.7) in each period $t$, and thus the number of independent equations, including (6.4.7), equals the number of variables. The unknown probabilities, $PR_{kij}$, are the steady state joint probabilities of the initial storage being in class $k$ and inflow being in class $i$ in period $t$. It is thus possible to obtain the marginal probabilities of storage and inflow as follows:

$$PS_{kt} = \sum_{l} PR_{kil} \quad \forall \; k, t \quad (6.4.8)$$

$$PQ_{it} = \sum_{l} PR_{lji} \quad \forall \; i, t \quad (6.4.9)$$

**6.4.4 Real-time Operation**

The SDP formulation discussed in the previous section assumes that the inflow during the period $t$ is known at the beginning of the period $t$ itself. In practice,
however, this will not be the case. Therefore, to apply the optimal steady state policy $l^*(k, i, t)$ for the period $t$ in real-time, an appropriate inflow-forecasting model will be necessary, which will enable a decision at the beginning of time period, $t$. Given such a forecasting model, the steady state policy determined by the SDP is used for real-time operation as follows: At the beginning of the current period $t$, the reservoir storage is noted. From the storage state discretization used in the SDP, the storage class interval $k$ is determined. The inflow forecasting model is used to get the forecasted inflow during the period $t$. Generally, the inflow forecasting models use the latest available inflow value (that is, the inflow during the previous period $t-1$, which will be known at the beginning of the current period $t$). From the inflow discretization scheme used in the SDP, the inflow class interval, $i$, corresponding to the forecasted inflow is determined. Knowing both $k$ and $i$ for the period $t$, the optimal end-of-the-period storage class interval is obtained from the steady state policy, $l^*(k, i, t)$.

The storage state representative value corresponding to this optimal storage class interval is $S_{k^*}^{i^*}$. The reservoir release is made to ensure an end-of-the-period storage, as close to $S_{k^*}^{i^*}$, as possible. A finer discretization of the storage and inflow state variables will improve the application potential of the steady state policy.

**Example 4.4** Obtain steady state policy for the following data, when the objective is to minimize the expected value of the sum of the square of deviations of release and storage from their respective targets, over a year with two periods. Neglect evaporation loss. If the release is greater than the release target, the deviation is set to zero.

For period 1

<table>
<thead>
<tr>
<th>$i$</th>
<th>$Q_i$</th>
<th>$k$</th>
<th>$S_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td>1</td>
<td>30</td>
</tr>
<tr>
<td>2</td>
<td>25</td>
<td>2</td>
<td>40</td>
</tr>
</tbody>
</table>

For period 2

<table>
<thead>
<tr>
<th>$i$</th>
<th>$Q_i$</th>
<th>$k$</th>
<th>$S_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>35</td>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>45</td>
<td>2</td>
<td>30</td>
</tr>
</tbody>
</table>

Target storage $T_s = 30$

Target release $T_r = 30$

$R_{i|j} = (R_{i|j} - T_r)^2 + (S_k - T_s)^2$

Inflow transition probabilities

<table>
<thead>
<tr>
<th>$i$</th>
<th>$j$</th>
<th>$i$</th>
<th>$j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>1</td>
<td>0.4</td>
</tr>
<tr>
<td>2</td>
<td>0.3</td>
<td>2</td>
<td>0.8</td>
</tr>
<tr>
<td>2</td>
<td>0.7</td>
<td>2</td>
<td>0.2</td>
</tr>
</tbody>
</table>
Model Development

Solution:

First the system performance measure, $B_{kil}$, is tabulated for all $k$, $i$, $l$, and $t$.

Period, $t = 1$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$S_i^j$</th>
<th>$Q_l$</th>
<th>$i$</th>
<th>$S_l^{ij}$</th>
<th>$E_{10}$</th>
<th>$R_{10}$</th>
<th>$(S_i^j - T_{j,l})^2$</th>
<th>$(R_{10} - T_{j,l})^2$</th>
<th>$B_{kil}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30</td>
<td>15</td>
<td>1</td>
<td>20</td>
<td>0</td>
<td>25</td>
<td>0</td>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>1</td>
<td>30</td>
<td>15</td>
<td>2</td>
<td>30</td>
<td>0</td>
<td>15</td>
<td>0</td>
<td>225</td>
<td>225</td>
</tr>
<tr>
<td>1</td>
<td>30</td>
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<td>1</td>
<td>20</td>
<td>0</td>
<td>35</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>30</td>
<td>25</td>
<td>2</td>
<td>30</td>
<td>0</td>
<td>25</td>
<td>0</td>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>15</td>
<td>1</td>
<td>20</td>
<td>0</td>
<td>35</td>
<td>100</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>15</td>
<td>2</td>
<td>30</td>
<td>0</td>
<td>25</td>
<td>100</td>
<td>25</td>
<td>125</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>25</td>
<td>1</td>
<td>20</td>
<td>0</td>
<td>45</td>
<td>100</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>25</td>
<td>2</td>
<td>30</td>
<td>0</td>
<td>35</td>
<td>100</td>
<td>0</td>
<td>100</td>
</tr>
</tbody>
</table>

The SDP iterations are shown in the following tables.

$n = 1, t = 2$

\[ f_2^1(k, i) = \min \left\{ B_{kil} \right\} \quad \forall, k, i \]

\[ \{\text{feasible } l\} \quad j \]

<table>
<thead>
<tr>
<th>$k$</th>
<th>$S_i^j$</th>
<th>$Q_l$</th>
<th>$i$</th>
<th>$S_l^{ij}$</th>
<th>$E_{10}$</th>
<th>$R_{10}$</th>
<th>$(S_i^j - T_{j,l})^2$</th>
<th>$(R_{10} - T_{j,l})^2$</th>
<th>$B_{kil}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td>35</td>
<td>1</td>
<td>30</td>
<td>0</td>
<td>25</td>
<td>100</td>
<td>25</td>
<td>125</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>35</td>
<td>2</td>
<td>40</td>
<td>0</td>
<td>15</td>
<td>100</td>
<td>225</td>
<td>325</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>45</td>
<td>1</td>
<td>30</td>
<td>0</td>
<td>35</td>
<td>100</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>45</td>
<td>2</td>
<td>40</td>
<td>0</td>
<td>25</td>
<td>100</td>
<td>25</td>
<td>125</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>35</td>
<td>1</td>
<td>30</td>
<td>0</td>
<td>35</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>35</td>
<td>2</td>
<td>40</td>
<td>0</td>
<td>25</td>
<td>0</td>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>45</td>
<td>1</td>
<td>30</td>
<td>0</td>
<td>45</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>45</td>
<td>2</td>
<td>40</td>
<td>0</td>
<td>35</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$n = 2, t = 1$

\[ f_2^1(k, i) = \min \left\{ B_{kil} + \sum_j P_{ij} f_1^j(l, j) \right\} \quad \forall, k, i \]

\[ \{\text{feasible } l\} \quad j \]

<table>
<thead>
<tr>
<th>$k$</th>
<th>$i$</th>
<th>$l$</th>
<th>$f_2^1(k, i)$</th>
<th>$l^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>125.00</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>25</td>
<td>125.00</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.00</td>
<td>0.00</td>
<td>1, 2</td>
</tr>
</tbody>
</table>

Example calculations for Stage 2:

$k = 1, i = 1, l = 1; B_{kil} + \sum_j P_{ij} f_1^j(l, j) = 25.0 + 0.50*125.0 + 0.50*100.0 = 137.5$

$k = 1, i = 1, l = 2; B_{kil} + \sum_j P_{ij} f_1^j(l, j) = 225.0 + 0.50*0.0 + 0.50*0.0 = 225.0$
Proceeding in a similar manner, further computations are shown as follows:

\[ \begin{array}{c|c|c|c|c|c|c}
 k & i & j & l & f^k_{ij}(k, i) & i^* \\
 \hline
 1 & 1 & 1 & 1 & 137.50 & 137.50 & 1 \\
 1 & 2 & 2 & 2 & 107.50 & 25.00 & 2 \\
 2 & 1 & 2 & 1 & 212.50 & 125.00 & 2 \\
 2 & 2 & 1 & 2 & 207.50 & 100.00 & 2 \\
 \hline
\end{array} \]

\[ f^3_{ij}(k, i) = \text{Min} \left\{ B_{kij} + \sum_j p^k_{ij} f^j_{ij}(l, j) \right\} \text{ for feasible } l \]

\[ \begin{array}{c|c|c|c|c|c|c|c}
 k & i & j & l & f^k_{ij}(k, i) & i^* \\
 \hline
 1 & 1 & 1 & 1 & 195.00 & 195.00 & 1 \\
 1 & 2 & 2 & 2 & 215.00 & 215.00 & 1 \\
 2 & 1 & 2 & 1 & 70.00 & 70.00 & 1 \\
 2 & 2 & 1 & 2 & 115.00 & 115.00 & 1 \\
 \hline
\end{array} \]

\[ f^4_{ij}(k, i) = \text{Min} \left\{ B_{kij} + \sum_j p^k_{ij} f^j_{ij}(l, j) \right\} \text{ for feasible } l \]

\[ \begin{array}{c|c|c|c|c|c|c|c}
 k & i & j & l & f^k_{ij}(k, i) & i^* \\
 \hline
 1 & 1 & 1 & 1 & 230.00 & 230.00 & 1 \\
 1 & 2 & 2 & 2 & 209.00 & 209.00 & 2 \\
 2 & 1 & 2 & 1 & 305.00 & 305.00 & 2 \\
 2 & 2 & 1 & 2 & 309.00 & 309.00 & 2 \\
 \hline
\end{array} \]

\[ f^5_{ij}(k, i) = \text{Min} \left\{ B_{kij} + \sum_j p^k_{ij} f^j_{ij}(l, j) \right\} \text{ for feasible } l \]

\[ \begin{array}{c|c|c|c|c|c|c|c}
 k & i & j & l & f^k_{ij}(k, i) & i^* \\
 \hline
 1 & 1 & 1 & 1 & 292.90 & 292.90 & 1 \\
 1 & 2 & 2 & 2 & 309.30 & 309.30 & 1 \\
 2 & 1 & 2 & 1 & 167.90 & 167.90 & 1 \\
 2 & 2 & 1 & 2 & 209.30 & 209.30 & 1 \\
 \hline
\end{array} \]

\[ f^6_{ij}(k, i) = \text{Min} \left\{ B_{kij} + \sum_j p^k_{ij} f^j_{ij}(l, j) \right\} \text{ for feasible } l \]

\[ \begin{array}{c|c|c|c|c|c|c|c}
 k & i & j & l & f^k_{ij}(k, i) & i^* \\
 \hline
 1 & 1 & 1 & 1 & 532.90 & 532.90 & 1 \\
 1 & 2 & 2 & 2 & 339.30 & 339.30 & 1 \\
 2 & 1 & 2 & 1 & 232.90 & 232.90 & 1 \\
 2 & 2 & 1 & 2 & 214.30 & 214.30 & 1 \\
 \hline
\end{array} \]
Model Development

The computations are terminated after this stage because it is verified that the annual system performance measure remains constant, (being nearly 96). The steady state policy is tabulated as follows:

Steady state policy for period 1

<table>
<thead>
<tr>
<th>$k$</th>
<th>$i$</th>
<th>$l^1$</th>
<th>$l^2$</th>
<th>$f_{k,i}^1$</th>
<th>$t^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>326.10</td>
<td>413.60</td>
<td>326.10</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>304.38</td>
<td>221.88</td>
<td>221.88</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>401.10</td>
<td>313.60</td>
<td>313.60</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>404.38</td>
<td>296.88</td>
<td>296.88</td>
<td>2</td>
</tr>
</tbody>
</table>

$\eta = 7 \ t = 2$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$i$</th>
<th>$l^1$</th>
<th>$l^2$</th>
<th>$f_{k,i}^2$</th>
<th>$t^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>388.57</td>
<td>628.57</td>
<td>388.57</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>405.26</td>
<td>435.26</td>
<td>405.26</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>263.57</td>
<td>328.57</td>
<td>263.57</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>305.26</td>
<td>310.62</td>
<td>305.26</td>
<td>1</td>
</tr>
</tbody>
</table>

$\eta = 8 \ t = 1$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$i$</th>
<th>$l^1$</th>
<th>$l^2$</th>
<th>$f_{k,i}^3$</th>
<th>$t^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>421.91</td>
<td>509.41</td>
<td>421.91</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>400.25</td>
<td>317.75</td>
<td>317.75</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>496.91</td>
<td>409.41</td>
<td>409.41</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>500.25</td>
<td>392.75</td>
<td>392.75</td>
<td>2</td>
</tr>
</tbody>
</table>

Verify that the expected annual system performance measure \[ f_{\text{sys}} (k, i) = \sum_{t=1}^{\eta} f_{k,i}^t \] remains constant for all $k$, $i$, and $t$ once the steady state is reached. In this example the expected annual performance is approximately 96 units (e.g. $f_{1,1}^1 - f_{1,1}^6 = 421.91 - 326.10 = 95.81$).
The steady state policy for a two-period problem is specified as follows:

\[ t = 1 \quad t = 2 \]

\[
\begin{array}{ccc}
  k & i & l' \\
  1 & 1 & 1 \\
  2 & 1 & 2 \\
  2 & 2 & 2 \\
\end{array}
\]

\[
\begin{array}{ccc}
  k & i & l' \\
  1 & 1 & 1 \\
  2 & 1 & 1 \\
  2 & 1 & 1 \\
\end{array}
\]

The inflow transition probabilities are as given in the following table.

\[
\begin{array}{ccc}
  t & 1 & 2 \\
  j & 1 & 1 \\
  1 & 0.5 & 0.5 \\
  2 & 0.3 & 0.7 \\
\end{array}
\]

\[
\begin{array}{ccc}
  t & 1 & 2 \\
  j & 2 & 1 \\
  1 & 0.4 & 0.6 \\
  2 & 0.8 & 0.2 \\
\end{array}
\]

Obtain the steady state probabilities of release, inflow, and storage.

**Solution:**

Computation of \( PR_{it} \)

Solving the following simultaneous equations will yield the values of \( PR_{it} \).

\[
PR_{it} = \sum_{k} \sum_{j} PR_{it} = \sum_{l} PR_{it} = \sum_{l} PR_{it} = 1 \quad \forall t
\]

\( t = 1; \)

\[
PR_{112} = PR_{113} * 0.5; \\
PR_{122} = PR_{111} * 0.5; \\
PR_{212} = PR_{211} * 0.3 + PR_{211} * 0.5 + PR_{221} * 0.3;
\]

[Note that \( t = 2 \) in period \( t = 1 \) results from three combinations of \((k, i, t)\), viz., (1, 2), (2, 1) and (2, 2). Hence the three terms in summation]

\[
PR_{222} = PR_{211} * 0.7 + PR_{211} * 0.5 + PR_{211} * 0.7; \quad (A)
\]

\( t = 2; \)

\[
PR_{111} = PR_{112} * 0.4 + PR_{122} * 0.8 + PR_{212} * 0.4; \\
PR_{121} = PR_{112} * 0.6 + PR_{222} * 0.2 + PR_{212} * 0.6; \\
PR_{211} = PR_{222} * 0.8; \\
PR_{221} = PR_{222} * 0.2; \quad (B)
\]

Additionally,

\[
PR_{111} + PR_{121} + PR_{211} + PR_{221} = 1; \\
PR_{112} + PR_{122} + PR_{212} + PR_{222} = 1; \quad (C)
\]
There is a total of 8 probabilities, $PR_{ki}$.

Taking any three of equations (A) for $t = 1$, any three equations (B) for $t = 2$, and the last two equations (C) (one for each period $t$), the set of 8 equations are solved for 8 values of $PR_{ki}$. The resulting $PR_{ki}$ values are shown in the following table.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$i$</th>
<th>$PR_{ki}$</th>
<th>$k$</th>
<th>$i$</th>
<th>$PR_{ki}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.284</td>
<td>1</td>
<td>1</td>
<td>0.142</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.346</td>
<td>2</td>
<td>1</td>
<td>0.284</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.086</td>
<td>2</td>
<td>2</td>
<td>0.432</td>
</tr>
</tbody>
</table>

Steady state probabilities of storage and inflow:

**Storage:**

$PS_{k_t} = \sum_i PR_{ki}$ \quad \forall k, t

$t = 1$

$PS_{11} = PR_{111} + PR_{121}$

$= 0.284 + 0.284$

$= 0.568$

$PS_{12} = PR_{112} + PR_{122}$

$= 0.142 + 0.142$

$= 0.284$

$t = 2$

$PS_{21} = PR_{211} + PR_{221}$

$= 0.346 + 0.086$

$= 0.432$

$PS_{22} = PR_{212} + PR_{222}$

$= 0.284 + 0.432$

$= 0.716$

**Inflow:**

$PQ_{i_t} = \sum_k PR_{ki}$ \quad \forall i, t

$t = 1$

$PQ_{11} = PR_{111} + PR_{121}$

$= 0.284 + 0.346$

$= 0.63$

$PQ_{12} = PR_{112} + PR_{122}$

$= 0.142 + 0.142$

$= 0.284$

$t = 2$

$PQ_{21} = PR_{211} + PR_{221}$

$= 0.284 + 0.086$

$= 0.37$

$PQ_{22} = PR_{212} + PR_{222}$

$= 0.142 + 0.432$

$= 0.574$
6.4.1 Solve the two-state, two-period SDP reservoir operation problem with the following data to obtain steady state release policy.

### Transition Probabilities

<table>
<thead>
<tr>
<th>Period $t = 1$</th>
<th>$Q^t$</th>
<th>$S^t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td>30</td>
</tr>
<tr>
<td>2</td>
<td>25</td>
<td>40</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Period $t = 2$</th>
<th>$Q^t$</th>
<th>$S^t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>35</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>45</td>
<td>30</td>
</tr>
</tbody>
</table>

Reservoir capacity = 60 units

\[ R_{\text{init}} = 1 \cdot R_{\text{end}} + |S^t| - T^t \]  

with $T^t = 55; T_s = 30$

6.4.2 The steady state release probabilities obtained from an SDP solution are given below, with notations followed in the text.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$i$</th>
<th>$P_R^{ki}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.112</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.053</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.144</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.532</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.080</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0.079</td>
</tr>
</tbody>
</table>

Obtain the steady state probabilities of inflows and storages.

6.4.3 The following inflow transition probabilities are given for a two-state, two-period SDP reservoir operation problem.

\[ \begin{align*}
& t = 1 & t = 2 \\
& 0.9 & 0.1 & 0.4 & 0.6 \\
0.5 & 0.5 & 0.2 & 0.8 \\
\end{align*} \]

The steady state policy obtained from the SDP solution is as given below:

\[ \begin{align*}
& t = 1 & t = 2 \\
& k & i & l^* & k & i & l^* \\
1 & 1 & 1 & 1 & 1 & 2 \\
1 & 2 & 1 & 1 & 2 & 2 \\
2 & 1 & 2 & 2 & 1 & 2 \\
2 & 2 & 1 & 2 & 2 & 1 \\
\end{align*} \]

Obtain the steady state probabilities for release, storage, and inflows from this solution.
REFERENCES


Further Reading

Applications

- Applications of Linear Programming
- Applications of Dynamic Programming
- Recent Modelling Tools
In this chapter we discuss some applications of linear programming to water resources problems, with a specific focus on irrigation.

7.1 IRRIGATION WATER ALLOCATION FOR SINGLE AND MULTIPLE CROPS

Most problems dealing with planning for irrigation schedules involve allocation of a known quantity of water across intraseasonal time periods, consisting typically of one week to ten days. Such problems may be posed as optimization problems and, with suitable simplifications, may be formulated as linear programming (LP) problems. In this section, we discuss such formulations, and provide indicative results of example applications. The concept of crop yield optimization is first discussed as a prerequisite for understanding the material covered subsequently in the chapter.

7.1.1 Crop Yield Optimization

The objective of an irrigation water management problem is to maximize a measure of crop production, taking into account the response of the crop to the amount of irrigation applied. To achieve this through a mathematical programming technique, crop production functions are normally used. Typical crop production functions (see, Doorenbos and Kassam, 1979 and Sarma, 2002 for a detailed discussion on crop production functions) relate the yield ratio (ratio of actual to maximum yield) of a crop to a function of the evapotranspiration ratio (ratio of actual to potential evapotranspiration) over the growth stages. At present, a single crop production function that is applicable to all crops, growing seasons, and climates is not available. We use here a simple additive form of production function to discuss formulation of the optimization problems. This is of the form
where \( N_g \) is the number of growth stages in the crop season.

\( g \) is the growth stage index

\( K_{yg} \) is the yield factor for the growth stage \( g \)

\( y \) is the actual yield of the crop

\( y_{\text{max}} \) is the maximum yield of the crop

AET is the actual evapotranspiration, and

PET is the potential evapotranspiration

The yield factor, \( K_{yg} \), indicates the sensitivity of the crop yield to the evapotranspiration deficit in growth stage \( g \). The soil moisture balance, which forms an important constraint in the optimization problems for irrigation water allocation, is schematically shown in Fig. 7.1, and is written for a crop \( c \) as

\[
\begin{align*}
\theta^{t+1}_c &= \theta^t_c + \Delta \theta^t_c - AET^t_c + \theta^t_c \Delta D_f^t + \theta^t_c \Delta \theta^t_c - DP^t_c \\
\end{align*}
\]

where \( \theta^t_c \) is the soil moisture of crop \( c \) at the beginning of the period \( t \), \( \Delta \theta^t_c \) is the root depth of crop \( c \) during period \( t \), RAIN is the effective rainfall (contribution of rainfall to soil moisture) in the command area in period \( t \), \( q^t_c \) is the irrigation application to crop \( c \) in period \( t \), \( AET^t_c \) is the actual evapotranspiration of crop \( c \) in period \( t \), \( \theta^t_f \) is the initial soil moisture in the soil zone into which the crop root extends at the beginning of period \( t + 1 \), and \( \Delta \theta^t_c \) is the deep percolation. The soil moisture values \( \theta^t_c \) and \( \theta^t_f \) are in units of depth per unit root depth and all other terms are in depth units.

![Schematic Diagram for Soil Moisture Balance](image)

The relationship between \( \text{AET}/\text{PET} \) ratio and the available soil moisture is approximated by a linear relationship, with \( \text{AET} = 0 \), when the available soil moisture is zero (corresponding to the actual soil moisture at wilting point) and \( \text{AET} = \text{PET} \) when the available soil moisture is equal to the maximum available soil moisture (corresponding to the actual soil moisture at field capacity). \( \theta_f \) and \( \theta_w \) are soil moisture at field capacity and wilting point, respectively, in depth per unit depth (of root zone) units. For use in linear models, the AET constraint is written as
Applications of Linear Programming

Consider the problem of allocation of a known amount of water available for the entire season among different intraseasonal periods of a crop. This problem may be formulated as an LP problem as follows:

**Max** \( \sum_{i=1}^{T} K_i \left( \frac{AET_i}{PET_i} \right) \)

Subject to

\[ AET_i \leq PET_i \quad \forall t \]

\[ AET_i \leq (\theta^0 + q^t + RAIN_i - AET_i - DP_i) \times \frac{PET_i}{(\theta_f - \theta_w)} \quad \forall t \]

\[ AET_i \leq PET_i \quad \forall t \]

\[ \theta^0 + q^t \geq M \times \beta \quad \forall t \]

\[ DP_i \leq M \times \beta \quad \forall t \]

\[ \theta_f \leq \theta^t \leq \theta_w \quad \forall t \]

\[ \sum_{i=1}^{T} q^t A \leq Q \]

with nonnegativity of all decision variables. \( \beta \) is a binary integer variable and \( M \) is a large number, both introduced to ensure that the deep percolation is nonzero only when the soil moisture is at field capacity. \( Q \) is the known amount of water (volume) available for the entire season, and \( q_i \) is the amount of water (depth) allocated in period \( t \), \( A \) is the crop area. The problem is formulated for a constant root depth so that the soil moistures \( \theta, \theta_f \) and \( \theta_w \) in the above formulation are in depth units.

**Example 7.1.1** Consider the following data for a single crop:
- Crop type: two seasonal (such as cotton grown in many parts of South India)
- Crop period: 6 months (18 intraseasonal time periods of 10 days each)
- Crop area: 3902.50 hectares
- Field capacity: 33.2\% (3.32 mm/cm)
- Wilting point: 16.5\% (1.65 mm/cm)
- Root Depth of the crop: 100 cm (assumed to be constant)

**Rainfall Data**

<table>
<thead>
<tr>
<th>Time periods (10 days)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Rainfall (in mm)</strong></td>
<td>0.5</td>
<td>0.0</td>
<td>0.0</td>
<td>120.0</td>
<td>2.0</td>
<td>0.0</td>
<td>1.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>150.0</td>
<td>7.0</td>
<td>3.0</td>
<td>0.0</td>
<td>6.7</td>
<td>0.0</td>
<td></td>
</tr>
</tbody>
</table>
Initial soil moisture is assumed to be just above the wilting point. When the LP problem is solved for various levels of available water, $Q$, the resulting allocations, variations of AET, and the soil moistures will be as shown in Figs. 7.2 to 7.4. The crop yield ratio (ratio of crop yield to maximum crop yield) resulting from the optimal allocations are given in Table 7.1. Such analyses are useful in irrigation scheduling of a known seasonal amount of available water.
Applications of Linear Programming

Table 7.1 Variation of Yield Ratio with Amount of Available Water (Q)

<table>
<thead>
<tr>
<th>SI No.</th>
<th>Q (Mm$^3$)</th>
<th>Yield Ratio$^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30</td>
<td>1.000</td>
</tr>
<tr>
<td>2</td>
<td>25</td>
<td>1.000</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>0.823</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>0.465</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>0.085</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>0.0</td>
</tr>
</tbody>
</table>

$^*$yield ratio = $y/y_m$ (see Eq. 7.1).
7.1.2 Irrigation Water Allocation to Multiple Crops

Consider the problem of water allocation in canal command areas, in which a known amount of water, \( Q_t \), \( t = 1, 2, \ldots, T \), is to be allocated to multiple crops in periods across their growing seasons. The LP model already discussed for single crop allocation may be modified as follows, with \( c \) indicating a crop, and \( t \) a time period:

\[
\text{Max} \quad \sum_{c=1}^{N} \sum_{t=1}^{T} K_{ct} \left( \frac{AET_c}{PET_c} \right) \\
\text{Subject to} \\
\theta_{ct}^* = \theta_{ct}^0 + q_c^t + \text{RAIN}_t - \text{AET}_c^t - \text{DP}_c^t \quad \forall c, t \\
\text{AET}_c^t \leq (\theta_{ct}^0 + q_c^t + \text{RAIN}_t - \theta \_a^0) \times \frac{\text{PET}_c}{(\theta_c - \theta \_a^0)} \quad \forall c, t \\
\theta_{ct}^0 \geq \theta_c \times \beta_c \quad \forall c, t \\
\text{DP}_c^t \leq M \times \beta_c \quad \forall c, t \\
\theta_c \leq \theta_{ct}^0 \leq \theta_c \quad \forall c, t \\
\sum_{c=1}^{N} q_c^t \leq Q \quad \forall t
\]

The notations used have the same meaning as before, with the subscript \( c \) denoting crop \( c \). \( N \) is the number of crops.

**Example 7.1.2** Consider an allocation problem for three crops. The crop calendar, available irrigation water, and rainfall in the command area are given in Fig. 7.5 and Table 7.2, for a season consisting of 18 ten-day periods. The crop areas are 3902.50 ha (cotton), 1977.43 ha (jowar) and 33.10 ha (groundnut). The PET values for these crops are computed from pan evaporation data (not given here). The problem is to allocate the known amount of water \( Q_t \), in period \( t \), to the three crops, such that the total (relative) crop yield is maximized at the end of the season. Note that the amount of water available in a period is a deterministic value, known in advance for all periods \( t \).
The field capacity is 33.20% and wilting point is 16.50%. The root depth is assumed to be 100 cm throughout the season for all crops. The problem is analyzed for various levels of available irrigation water with the available amount of water given in Table 7.2, taken as the maximum amount $Q$. Figures 7.6 to 7.8 show variations of allocations to the three crops with various levels of available water.

**Table 7.2** | Rainfall and Available Irrigation Supply

<table>
<thead>
<tr>
<th>Time periods (10 days)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rainfall (Rain t (mm))</td>
<td>0.5</td>
<td>0.0</td>
<td>0.0</td>
<td>120.0</td>
<td>2.0</td>
<td>2.0</td>
<td>0.0</td>
<td>1.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Available irrigation supply ($Q_t$) Mean</td>
<td>3.65</td>
<td>3.65</td>
<td>3.65</td>
<td>5.70</td>
<td>3.70</td>
<td>3.50</td>
<td>3.20</td>
<td>3.20</td>
<td>3.50</td>
<td>3.50</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time periods (10 days)</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rainfall (Rain t (mm))</td>
<td>0.0</td>
<td>1.0</td>
<td>150.0</td>
<td>7.0</td>
<td>3.0</td>
<td>0.0</td>
<td>6.7</td>
<td>0.0</td>
</tr>
<tr>
<td>Available irrigation supply ($Q_t$) Mean</td>
<td>3.70</td>
<td>3.70</td>
<td>5.80</td>
<td>5.60</td>
<td>4.00</td>
<td>4.00</td>
<td>5.00</td>
<td>3.00</td>
</tr>
</tbody>
</table>
Fig. 7.7 | Variation of Water Allocation With Water Availability (Q) for Jowar

Fig. 7.8 | Variation of Water Allocation with Water Availability (Q) for Groundnut
Allocations obtained from the LP model account for competition among crops for water through the crop sensitivity factors $K_c$. The optimal allocations in a period depend on a number of factors such as water available, time of season, soil moistures of individual crops in the period, potential evapotranspiration of the crops, rainfall in the period, and crop areas. Note that in this example, the water available in intraseasonal periods across the crop season is assumed to be known. In planning reservoir releases for irrigation, however, the water available for irrigation itself becomes a decision variable, along with cropwater allocations, as shown in Secs. 7.3 and 7.4.

### 7.2 MULTIRESERVOIR SYSTEM FOR IRRIGATION PLANNING

A river basin is considered for expanding its irrigation operations in its relatively less developed upper basin, without detriment to the irrigation needs of the lower basin, which is fully developed.

A four-reservoir system in the upper basin is considered for development with given reservoir capacities. The crops proposed to be grown and the land area available for cultivation in the command area of each reservoir are given. Irrigation requirements of each crop per unit area are worked out at each site and are known.

The four-reservoir system comprising the upper basin is modelled with monthly time periods by using LP to determine the crops to be irrigated and the extent of irrigation at each reservoir subject to land, water, and downstream release constraints. Under conditions of limited water supply, there are two options of irrigation considered: one is to provide intensive irrigation to those crops that yield higher economic returns (these are also the crops that need relatively more water) and leave the others unirrigated; the other is to allocate the available water to crops over a wider area to provide extensive irrigation. Accordingly, two objectives are studied: one of maximizing the net economic benefits (from both irrigated as well as unirrigated cropped areas at all reservoirs) and the other of maximizing the total irrigated cropped area at all the reservoirs in the upper basin. These two conflicting objectives are studied for analyzing the tradeoff between them from the point of view of multiobjective planning, using the constraint method (Section 4.2.2).

The model decides the crops to be irrigated and the extent of irrigation. If a crop is considered at all for irrigation, under either option mentioned above, it receives its full water requirements in all periods of the crop season.

The four-reservoir system in the upper basin is shown in Fig. 7.9. There are two reservoirs $B$ and $D$ on the main river, and the other two, $A$ and $C$, on the tributaries. There is existing irrigation at reservoirs $B$ and $D$. Expansion of irrigation at $B$ and new cropped areas at $A$ and $C$ are proposed.
Applications

The reservoir capacities and land area proposed for irrigation are as shown in Table 7.3.

<table>
<thead>
<tr>
<th>Reservoir</th>
<th>Drainage area, km²</th>
<th>Capacity, Mm³</th>
<th>Land area proposed for irrigation, ha</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A</td>
<td>5200</td>
<td>962.77</td>
</tr>
<tr>
<td>2</td>
<td>B</td>
<td>9666</td>
<td>1,268.59</td>
</tr>
<tr>
<td>3</td>
<td>C</td>
<td>6693</td>
<td>453.07</td>
</tr>
<tr>
<td>4</td>
<td>D</td>
<td>11993</td>
<td>2467.60</td>
</tr>
</tbody>
</table>

*This is in addition to the existing irrigation at Reservoir B.

The Model

A linear programming model is formulated with monthly time periods to determine the crops and crop areas to be irrigated at each of the sites A, B, and C, subject to minimum downstream releases from reservoir D to meet the existing irrigation requirement in the lower basin. The model considers the option of irrigating or not irrigating any or all of the cropped area under each crop, for each objective.

Objective 1 Maximize the total net economic benefits from all crops in the upper basin.

The objective function is written as

$$\max \sum_{j} \sum_{i} \alpha_{ij} I_{ij} + \sum_{i} \sum_{j} \beta_{ij} U_{ij}$$

$\alpha_{ij}$ net benefits at site $j$ per unit area of irrigated crop $i$, $j = 1$, reservoir A; $j = 2$, reservoir B; and $j = 3$, reservoir C;

$\beta_{ij}$ net benefits at site $j$ per unit area of unirrigated crop $i$;

$I_{ij}$ irrigated area at site $j$ under crop $i$;

$U_{ij}$ unirrigated area at site $j$ under crop $i$; and

$M_{ij}$ total number of crops considered at site $j$. 

Fig. 7.9 Schematic Diagram of the River Basin under Study
Applications of Linear Programming

**Objective 2** Maximize total irrigated area from all crops in the upper basin.

\[
\max \sum_{j=1}^{M} \sum_{i=1}^{I} I_{ij}
\]

**Constraints**

**Land Allocation Constraints** The same land should be made available for a given crop throughout the growing season.

For all the growing months \( m \) of crop \( i \) at site \( j \)
\[
I_{ijm} = I_{ij} \quad \text{for all } m,
\]
where \( I_{ij} \) is the irrigated area at site \( j \) under crop \( i \) in period \( m \).

For all the growing months \( m \) of crop \( i \) at site \( j \),
\[
U_{ijm} = U_{ij} \quad \text{for all } m,
\]
where \( U_{ij} \) is the unirrigated area at site \( j \) under crop \( i \) in period \( m \).

**Storage—Continuity Equations** The equations for the four sites \( j = 1, 2, 3, 4 \) are written as follows:

These constraints are valid for all time periods, \( m = 1, 2, \ldots, 12 \).

For \( j = 1 \)
\[
S_{1m} - S_{1m+1} - \sum_{i=1}^{I} p_{im}I_{im} - R_{1m} = E_{1m} - F_{1m}
\]

For \( j = 2 \)
\[
S_{2m} - S_{2m+1} + R_{2m} - \sum_{i=1}^{I} p_{im}I_{im} - R_{2m} = E_{2m} - F_{2m} + W_{2m}
\]

For \( j = 3 \)
\[
S_{3m} - S_{3m+1} - \sum_{i=1}^{I} p_{im}I_{im} - R_{3m} = E_{3m} - F_{3m}
\]

For \( j = 4 \)
\[
S_{4m} - S_{4m+1} + R_{2m} + R_{3m} - R_{4m} = E_{4m} - F_{4m} + W_{4m}
\]

where

- \( S_{jm} \) storage at site \( j \) at the beginning of month \( m \);
- \( p_{im} \) water diversion requirement of crop \( i \) (in depth units) at \( j \) during its growing month \( m \), and equal to zero for the nongrowing months;
- \( R_{jm} \) d/s release from the reservoir at site \( j \) during the month \( m \);
- \( E_{jm} \) Reservoir evaporation at site \( j \) during the month \( m \);
- \( F_{jm} \) mean inflow (unregulated) at site \( j \) during month \( m \); and
- \( W_{jm} \) water diversion requirement at site \( j \) for all existing crops during month \( m \) (this term will appear only for \( j = 2 \), and for \( j = 4 \), as per existing irrigation requirements. Additional irrigation to ten proposed crops, \( i \), at site \( j = 2 \) is considered).
Land Area Constraints  The total cropped area (irrigated and/or unirrigated) in each month is less than the possible maximum at each site.

\[ \sum_{j=1}^{M} (U_{jm} + U_{jm}) \leq L_j \quad j = 1, 2, 3 \quad m = 1, 2, ..., 12 \]

where \( L_j \) = land available under the command of the reservoir at site \( j \).

For a given crop calendar, the total number of governing constraints at each site may be less than 12 because some constraints may be repetitive or may form a subset of other constraints. In a month where one crop ends and another begins, the area of the crop ending in that month is not included in the constraint to permit the possible use of the land for both crops in that month.

Downstream Release Constraints  The release from the reservoir at site 4, \( R_{4m} \), should be greater than the minimum downstream release specified by \((DR)_{m} \), which is arrived at on the basis of the requirements for existing irrigation below reservoir \( D \). Then

\[ R_{4m} \geq (DR)_{m} \quad \forall \ m \]

Capacity Constraints  The storage in each reservoir is limited to its capacity (live storage) at all times.

\[ S_{jm} \leq S_{j\text{max}} \quad \forall \ j, m \]

where \( S_{j\text{max}} \) is the live storage capacity of the reservoir at site \( j \).

Crop Irrigated Unirrigated
---
Rice 2950 1525
Ragi/Jowar 2450 1450
Maize 2300 1300
Groundnut 2400 1100
Potato 2500 2380
Wheat 1900 825
Soyabeans 2800 1300
Pulses/Vegetables 600 360
Safflower 375 225

*Obtained by deducting estimated cost of fertilizer from produce value at wholesale prices.
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Table 7.5 Minimum Downstream Release Requirements at Reservoir D (Mm³)

<table>
<thead>
<tr>
<th>Month</th>
<th>Release</th>
<th>Month</th>
<th>Release</th>
</tr>
</thead>
<tbody>
<tr>
<td>June</td>
<td>570.0</td>
<td>Dec.</td>
<td>0</td>
</tr>
<tr>
<td>July</td>
<td>1668.5</td>
<td>Jan.</td>
<td>329.2</td>
</tr>
<tr>
<td>Aug.</td>
<td>1360.8</td>
<td>Feb.</td>
<td>569.5</td>
</tr>
<tr>
<td>Sept.</td>
<td>775.9</td>
<td>Mar.</td>
<td>708.6</td>
</tr>
<tr>
<td>Oct.</td>
<td>0</td>
<td>Apr.</td>
<td>140.0</td>
</tr>
<tr>
<td>Nov.</td>
<td>0</td>
<td>May</td>
<td>0</td>
</tr>
</tbody>
</table>

Solution

Objective 1 Maximization of net economic benefits (annual): The optimal cropping for this case is shown in Table 7.6. The table shows what crops are to be grown, whether they should be irrigated or not, to what extent they should be grown in each of the areas, and the growing seasons. The maximized net benefits and the corresponding values of the total diversion and the irrigated cropped area in the upper basin are also given in the table.

Table 7.6 Solution for Maximum Net Benefits

<table>
<thead>
<tr>
<th>Crop</th>
<th>Season</th>
<th>IU</th>
<th>Cropped Area, 100 ha</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rice</td>
<td>Jun–Nov</td>
<td>I</td>
<td>1244.5</td>
</tr>
<tr>
<td>Ragi/Jowar</td>
<td>Jun–Oct</td>
<td>I</td>
<td>1244.5</td>
</tr>
<tr>
<td>Soyabean</td>
<td>Nov–Feb</td>
<td>I</td>
<td>557.7</td>
</tr>
<tr>
<td>Potato</td>
<td>Dec–Mar</td>
<td>U</td>
<td>1931.3</td>
</tr>
<tr>
<td>Mulberry</td>
<td>Perennial</td>
<td>I</td>
<td>207.0</td>
</tr>
<tr>
<td>Rice</td>
<td>Jun–Nov</td>
<td>I</td>
<td>572.0</td>
</tr>
<tr>
<td>Soyabean</td>
<td>Nov–Feb</td>
<td>I</td>
<td>572.0</td>
</tr>
<tr>
<td>Potato</td>
<td>Nov–Feb</td>
<td>U</td>
<td>878.0</td>
</tr>
</tbody>
</table>

Maximum net benefits = Rs 2084.709 million; total irrigated cropped area = 527,570 ha (exclusive of fixed mulberry crop area of 20,700 ha). *I* = irrigated, *U* = unirrigated.

Objective 2 Maximization of irrigated cropped area (ICA): The solution obtained for this case is given in Table 7.7. The maximum net benefits (Objective 1) are Rs 2084.7 million and the corresponding irrigated area is 527,520 ha. The maximum irrigated cropped area (Objective 2) is 755,600 ha, an increase of 43%, and the corresponding net benefits are Rs 1659.6 million, a reduction of 20% when compared to the results of maximizing Objective 1. In this case, increase in the cropped area was made possible by replacing the rice and soyabean (November–February) crops with less water-requiring crops such as ragi, jowar, wheat, vegetables, and pulses. When the irrigated cropped area is maximized, the model chooses...
crops needing relatively lower water requirement, as the total amount of water available is limited. These are also the crops that yield lower economic benefits.

**Summary of Results**

<table>
<thead>
<tr>
<th>Variable maximized</th>
<th>Net Benefits</th>
<th>Irrigated Cropped Area (ICA)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Net benefits (10^6 Rs 1970–72)</td>
<td>2084.709*</td>
<td>5275.700*</td>
</tr>
<tr>
<td>Irrigated cropped area (100 ha)</td>
<td>1659.623</td>
<td>7556.00*</td>
</tr>
</tbody>
</table>

(*maximum value)

**A Multiobjective Approach** As the two objectives are conflicting with each other, multiobjective analysis deals with finding the tradeoff between net income and irrigated cropped area at different levels of development. Figure 7.10 shows the efficiency frontier or the transformation curve for these outputs. Each of the plotted points in the figure is obtained by maximizing the net benefits for a fixed value of the irrigated cropped area, using constraint approach (Section 4.2.2), and corresponds to a plan of cropping such as the one given in Table 7.6. Any point on the curve defined by these points is Pareto admissible.

**Implicit Tradeoff** A total irrigated cropped area of 576,800 ha. (at reservoirs A, B, and C) is projected for development by the concerned planning agency, based on rainfall distribution and local conditions. It is interesting to see that
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the proposed area lies within the range of values obtained for the two objectives (between 527,570 and 755,600 ha.). A chance constrained form of the model may be used to determine the results for inflows at a specified reliability level, though the study reported herein does not attempt it.

A tradeoff between the net benefits and the irrigated cropped area can be found from the curve above (Fig. 7.10) by determining the slope of the curve at a given value of the irrigated area. For the projected level of 576,800 ha., the tradeoff works out to Rs 360 per ha. This means that the amount of net benefits foregone for an increase in the irrigated cropped area (at the level of projection) is Rs. 360 per hectare, subject to the assumptions and limitations of the model.

For details of the study, refer to Vedula and Rogers (1981).

7.3 RELIABILITY CAPACITY TRADEOFF FOR MULTICROP IRRIGATION

In this application, the formulation runs along similar lines to the problem discussed in Section 6.3.1, except that in the present case we discuss how the reservoir capacity tradeoff with reliability can be arrived at in the case of multiple crops, taking into account the soil moisture balance in the root zone of individual crops (as against lumped demands considered earlier).

A single reservoir is considered for multicrop irrigation. Data on crops, soils, and potential evapotranspiration of each crop are known. A reliability-based reservoir-irrigation model is formulated to determine (i) minimum size of the reservoir needed to meet crop irrigation requirements at a specified level of reliability, and (ii) maximum reliability of meeting irrigation requirements associated with a given reservoir capacity, using chance constrained linear programming. Reservoir inflow is considered random, and rainfall in the command area is considered deterministic. The model takes into account water allocation to multiple crops, soil moisture balance in the root zone of the cropped area, heterogeneous soil types, and crop root growth with time, and
integrates the crop water allocation decisions with release decisions at the reservoir. The crops and the crop calendar are assumed fixed. Fortnightly irrigation decisions are considered within the crop season.

**The Model**

The salient features of model formulation not covered in Section 6.3.1 are covered in detail here.

**Reservoir Release Policy and Storage Continuity**

The reservoir release policy in terms of irrigation water allocations to individual crops and the storage continuity relationships are written as follows.

The reliability constraint (Section 6.3.1) is written here as:

\[
\text{Prob} \left( R_t \geq \frac{1}{\eta} \sum c_t \left( A_c^t \right) \right) \geq \alpha \quad \text{for all } t.
\]

where
- \( R_t \): release from the reservoir in period, \( t \)
- \( \eta \): a parameter equal to the ratio of the total crop water allocation at the plant level to the reservoir release (to account for conveyance and other losses).
- \( c_t^x \): irrigation allocation to crop \( c \) in period \( t \), in depth units
- \( A^t_c \): area of crop \( c \) in area units
- \( \alpha \): specified reliability level.

**Soil Moisture Balance**

Soil moisture balance equation for a given crop \( c \) in period \( t \) is expressed as (following Section 7.1.1),

\[
SM_t^c = \Delta R + \Delta RAIN_t + FC^c \left( D_{t+1}^c - D_t^c \right) - PET_t^c - DP_t^c = SM_{t+1}^c D_{t+1}^c
\]

where \( SM_t^c \) is the soil moisture at the beginning of the period \( t \) for crop \( c \) in depth units per unit root depth, \( D_t^c \) is the average root depth during period \( t \) for crop \( c \) in depth units, \( \Delta R \) is the irrigation allocation in period \( t \) for crop \( c \) in depth units, \( \Delta RAIN_t \) is the rainfall in period \( t \) in depth units, \( FC^c \) is the soil moisture at field capacity of the soil for crop \( c \) in depth units per unit root depth, \( PET_t^c \) is the potential evapotranspiration in period \( t \) for crop \( c \) in depth units, and \( DP_t^c \) is the deep percolation in period \( t \) for crop \( c \) in depth units. It is assumed that all of the rainfall infiltrates into the soil. It is implied in this constraint that \( \Delta R \) is drawn out of the reservoir release to meet the full requirement of \( PET_t^c \) of the crop. Values of \( \Delta R \) and release \( R_t \) are, however, controlled by the reliability criterion.

**Soil Moisture**

It is assumed that

(i) in the beginning of (the first period of) the season soil moisture is at at field capacity for all crops, \( SM_1^c = FC^c \),

(ii) the soil moisture in the incremental crop root depth at the end of the period is at field capacity, and
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(iii) soil moisture is bounded by its value at the wilting point ($WP_c$) and at field capacity ($FC_c$).

Potential Evapotranspiration

Estimation of the $PET_c$ for crop $c$ in a particular period $t$ is based on reference evaporation in period $t$, ($ET_{o_t}$), and crop factor $k_t^c$ for crop $c$ in period $t$.

$$PET_c t = k_t^c (ET_{o_t})$$

The crop factor is a function of the crop growth stage. Values of $k_t$ for all $t$ in a given growth stage are assumed to be equal to the crop factor for that growth stage.

Deep Percolation

Deep percolation is suitably penalized in the objective function so that it occurs only when the soil moisture at end of the period is at field capacity.

Deterministic Equivalent

The deterministic equivalent of the reliability constraint is written using a linear decision rule as in Section 6.3.1.

$$[(1 + a_t)b_t - (1 - a_t)b_{t-1} + I_t + D_t] \leq F^{-1}(1 - \alpha_t)$$

where $F^{-1}(1 - \alpha_t) = Q^{(1-\alpha_t)}$ = inflow in period $t$, with a probability of $(1 - \alpha_t)$, or an exceedance probability of $\alpha_t$.

Other Constraints

In order to ensure that the soil moisture is brought to field capacity every period, enabling extraction of water by the plants at their full demand at $PET_c$ (this assumes that actual evapotranspiration equals $PET$ only when the soil moisture is at field capacity in the light of the soil moisture balance equation written earlier), $x_t^c$ is constrained, as

$$SM_t^c D_t^c + x_t^c + RAIN_t \geq FC_t^c D_t^c$$

for all $t$.

The total irrigation allocation in any period is to be less than the canal capacity, CC, in each period.

$$\sum (x_t^c A^c) \leq CC$$

for all $t$.

where CC is the canal capacity given.

Objective Function

The objective function used is to minimize the active storage capacity $K$, with a penalty for deep percolation. Note that this technique of imposing penalty is different from the formulation in Sec 7.1.1, in which integer variables were used. Deep percolation will then occur only when it must, i.e. only when the end-of-period soil moisture is at field capacity. Thus the objective function is written as

$$\text{Minimize } [K + M \sum_t^{t'} DP_t^c]$$

where $M$ is an arbitrarily large value.
The model is applied to Ghataprabha reservoir in Karnataka to determine the minimum reservoir capacity for different levels of reliability. The live capacity of the reservoir is 1389 Mm$^3$. It is assumed that the reservoir releases water to a composite command comprising the total irrigated area by the left and right bank canals (which together have a total discharge capacity of 135.95 m$^3$/sec). This is done because of the model’s limitation in dealing with two different random releases into the two canals (left bank canal and right bank canal), both of which share the same source of supply (reservoir storage). Inflow and rainfall data of 35 years are used. Average rainfall in the command area is computed using Theissen weights.

The crops considered are hybrid maize (95,220 ha) and groundnut (31,740 ha) in the kharif season; and hybrid maize (63,480 ha), wheat (47,610 ha), and pulses (15,870 ha) in the rabi season. Field capacity of soil is 0.3 cm/cm and the wilting point is 0.1 cm/cm.

The maximum attainable reliability with existing live capacity of 1389 Mm$^3$ is determined to be 0.55 or 55%; and the maximum possible reliability, for the given inflow data, is 80%, for which the minimum required capacity is 1611 Mm$^3$. Reliability higher than this results in an infeasible solution. This analysis shows that, for the reservoir under study, the maximum attainable reliability is limited by the existing capacity.

The relationship between reliability and minimum required active reservoir capacity is plotted in Fig. 7.11. From the curve, the minimum required capacity for a specified reliability level (within the feasible limit), or the reliability associated with a given capacity, may be obtained.

![Reliability-Capacity Tradeoff Curve](image)

The details of the application are given in Vedula and Sreenivasan (1994).

### 7.4 Reservoir Operation for Irrigation

Figure 7.12 shows components of a typical surface water irrigation system. In modelling for integrated reservoir operation, the conveyance system is treated...
as a lumped system and all losses are accounted through a loss term in the water balance. The irrigation release decisions at the reservoir are to be made for short time intervals such as a week, ten days or, at the most, two weeks. Mathematical models which aid decisions over larger time intervals such as a month or a season are therefore inadequate, as they do not take into account the variability in irrigation demand within these time intervals. Mathematical models have therefore been developed to determine a long-term operating policy, considering the intraseasonal irrigation demands of the crops, and competition among them in the face of a deficit supply (e.g. Dudley, 1988; Vedula and Mujumdar, 1992).

7.4.1 Short Term Reservoir Operation for Irrigation

A short-term reservoir operation typically has an operating horizon of one year. The problem may be formulated as an LP problem with deterministic inflows. The inflows are obtained from an appropriate inflow forecasting model, such as the Auto Regressive Moving Average (ARMA) model. The LP model is formulated to be solved from the current period \( t_p \) in real time to the last period \( T \) in the year. The objective is to maximize a measure of crop yield, which may be expressed as minimization of weighted evapotranspiration deficit.

**Objective Function**

\[
\text{Min} \sum_{c=1}^{N_t} \sum_{t=1}^{T} P_c + K_{c} (1 - \frac{AET}{PET})
\]  

(7.4.1)

where \( t_p \) is the current period in real time, \( T \) is the last period in the year, \( N_t \) is the number of crops present in period \( t \), \( c \) is the crop index, \( AET_c \) is the actual evapotranspiration, \( PET_c \) is the potential evapotranspiration, \( K_{c} \) is the yield sensitivity factor of crop \( c \) in period \( t \), and \( P_c \) is a measure of production for crop \( c \) actually realized up to the beginning of the period \( t \). It may be noted that as the model is applied in real time, the actual value of the crop production function up to the beginning of the current period may be quite different from that predicted when the model was applied in the previous period. The term \( P_c \)
is the value of the second term of the objective function actually realized from the first period in the year up to the beginning of the current period \( t_p \). That is,

\[
P_p = \sum_{t=1}^{t_p-1} K_t (1 - \text{PET}_t/\text{PET}_t^o)
\]

Addition of this term in the objective function ensures that at every period, the allocations are made such that the crop yield in the entire year is maximized, using the latest available information on the state of the system. In the dynamic programming model discussed in the next chapter (Sec. 8.3), this feature is achieved by introducing an additional crop production state variable, \( c \) for each crop \( c \).

**Constraints**

**Reservoir Storage Continuity** The reservoir storage continuity is written, taking into account the storage-dependent evaporation losses, as

\[
(1 + a_t)S_{t+1} = (1 - a_t)S_t + Q_t - R_t - O_t - A_0 \epsilon_t \quad t = t_p, t_{p+1}, \ldots, T \quad (7.4.2)
\]

where \( a_t = A_0 \epsilon_t / 2 \). \( A_0 \) is the water spread area corresponding to the dead storage level, \( \epsilon_t \) is the evaporation rate in period \( t \), \( S_t \) is the storage at the beginning of period \( t \), \( Q_t \) is the forecasted inflow during period \( t \), \( R_t \) is the release during period \( t \), and \( O_t \) is the overflow during period \( t \). Note that while solving the model the term \( O_t \) should be penalized in the objective function to ensure that it takes nonzero values only when the reservoir storage exceeds the capacity.

The reservoir storage at the beginning of the current period \( t_p \) is known.

\[
S_{t_p} = S_0 \quad (7.4.3)
\]

where \( S_0 \) is the known live storage of the reservoir at the beginning of the period. In case of over-year storage, the end condition on the storage also gets fixed, and an appropriate constraint is included to force the condition that the storage at the end of the last period \( T \) should be at least equal to the specified over-year storage \( S_o \).

\[
S_{T+1} \geq S_o \quad (7.4.4)
\]

**Water Available for Irrigation** The reservoir release undergoes conveyance, application, and other losses. The water actually applied for irrigation of crops must therefore take into account these losses. When a release \( R_t \) is made at the reservoir in a period \( t \), the water available for allocation among crops is given by

\[
X_t = \eta R_t \quad t = t_p, t_{p+1}, \ldots, T \quad (7.4.5)
\]

where \( \eta \) is the conveyance efficiency accounting for all losses in the release.

**Soil Moisture Balance** The soil moisture at the beginning of the current period, \( t_p \), is known for all crops. Starting with this known soil moisture, the soil moisture at the beginning of all subsequent periods up to the end of the year are computed by the soil moisture continuity, given by

\[
\theta_t = \theta_{t-1} + \text{RAIN}_t + X_t - \text{PET}_t + B_0(D_t^{t+1} - D_t^o) - Dp_t^o \quad \forall c, t \quad (7.4.6)
\]
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where \( \theta^0_t \) is the soil moisture of crop \( c \) at the beginning of the period \( t \), \( D^f_t \) is the root depth of crop \( c \) during period \( t \), \( RAIN_t \) is the rainfall in the command area in period \( t \) (assuming all rainfall is available to the crop), \( \chi^f_t \) is the irrigation application to crop \( c \) in period \( t \), \( AET^f_t \) is the actual evapotranspiration of crop \( c \) in period \( t \), \( \theta_0 \) is the initial soil moisture in the soil zone into which the crop root extends at the beginning of period \( t + 1 \), and \( Dp^f_t \) is the deep percolation.

The soil moisture \( \theta_0 \) is assumed to be known, and in the model application it is taken to be equal to the field capacity. Sensitivity of the system performance with variation in \( \theta_0 \) is discussed in Mujumdar and Vedula (1992). The soil moistures \( \theta^0_t \), irrigation allocations \( \chi^f_t \), and the actual evapotranspirations \( AET^f_t \) are all decision variables. The relationship between the \( AET/PET \) ratio and the available soil moisture is approximated by a linear relationship, with \( AET = 0 \), when the available soil moisture is zero (corresponding to the actual soil moisture at wilting point) and \( AET = PET \) when the available soil moisture is equal to the maximum available soil moisture (corresponding to the actual soil moisture at field capacity). This condition is written as

\[
AET^f_t \leq \left( \frac{\theta_0^f \left(D^f_t + \sum_{c'} I^f_{c'} + \chi^f_t \right) - \theta_0^f D^f_t}{(\theta_0^f - \theta_0^f)} \right) PET^f_t \quad \forall c, t \quad (7.4.7)
\]

and

\[
AET^f_t \leq PET^f_t \quad \forall c, t \quad (7.4.8)
\]

The constraint (7.4.7) is necessary along with (7.4.8) to restrict the maximum value of the actual evapotranspiration to the potential evapotranspiration. The denominator in (7.4.7) is the maximum available soil moisture of crop \( c \) in period \( t \) and the term, \( (\theta_0^f D^f_t + \sum_{c'} I^f_{c'} + \chi^f_t) \), is the actual available soil moisture after the addition of rainfall and irrigation application. The crop root depth in period \( t \) is assumed to be known, and in model application an appropriate root depth model may be used.

**Allocation Limit** The total water available for irrigation \( X_t \) corresponding to the release \( R_t \) must equal the water actually allocated to the crops. That is, \( X_t = \sum_{c} X^t_c \quad \forall t \) \( (7.4.9) \)

where \( A_c \) is the area of crop \( c \) under irrigation. The upper limit on the soil moisture \( \theta^f_t \) is the field capacity. Any moisture in excess will go out of the root zone as deep percolation, which is ensured through the soil moisture balance equation.

\[
\theta^0_t \leq \theta_t^f \quad \forall c, t \quad (7.4.10)
\]

The LP model given by (7.4.1) to (7.4.10) is solved from the current period \( t_p \) in real time up to the last period \( T \) in the year. The reservoir storage \( S_{tp} \) at the beginning of the period \( t_p \), the actual inflow \( I_{tp-1} \) during the previous period \( t_p -1 \), and the soil moistures \( \theta^0_t \) at the beginning of the period \( t \) for all crops \( c \)
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are known. Inflow forecasts $Q_t$ for the periods starting from the current period $t_p$ to the last period $T$ are obtained by a suitable one-step-ahead-inflow-forecasting model (e.g. a model of the Auto Regressive Moving Average, ARMA, family) that uses the latest available historical inflow, $I_{t_p-1}$. The rainfall inputs may be provided in a similar way, although in monsoon climates with low coefficient of variation for rainfall in individual periods, mean rainfall values may be used as a deterministic input.

The LP model is applied to the case study of the Malaprabha reservoir. For the case study, water deficit exists mainly in the rabi season, since the kharif crops are supported to a large extent by the monsoon rains. Table 7.8 gives a typical output from the model run for the period 16 (first period in the rabi season). The results presented in the table are obtained by one run of the model with the current period, $T_p$, set to 16. The inputs required for this run are the storage at the beginning of the period, inflow during the previous (i.e. 15th) period, and the soil moistures of the crops at the beginning of period 16. As seen from Table 7.8, the two-seasonal crop cotton is already in the advanced growing season at the beginning of period 16, whereas the other crops are just at the beginning of the first growth stage. The soil moisture of cotton and the initial reservoir storage shown in the table are the values obtained by real-time simulation till the end of the 15th period for the particular year. The soil moistures of all other crops are assumed to be at field capacity, since the root depths are very small and, consequently, the amount of water, which is assumed available at the time of sowing, required to keep the root zone at field capacity is very low. The inflow values shown in the table are all forecasted inflows from an AR(1) model which uses the actual inflow, 11.32 Mm$^3$ for the period 15 as an input. In real time, only the release and allocation decisions for the current period, i.e. period 16, are meant to be implemented. The reservoir storage and the soil moistures are updated at the end of the period with the actual values of inflow and rainfall for the optimal release and allocation decisions.

7.5 RESERVOIR OPERATION FOR HYDROPOWER OPTIMIZATION

This section presents an application of linear programming formulation for the operation of a single reservoir for irrigation and hydropower. The purpose of the model is to maximize hydropower production, subject to satisfying irrigation demands at a specified reliability level. A piecewise linearization technique for the nonlinear power function is used in this application to enable use of linear programming as the optimizing tool. Details can be found in Sreenivasan and Vedula (1996).

The reservoir system consists of a reservoir with irrigation canals (on both sides of the river) leading to the irrigated area, and a riverbed powerhouse located at the bed of the river. Water drawn into the canals exclusively for irrigation incidentally produces a small amount of hydropower in the powerhouses located along the canals. Power is produced out of the water released
downstream of the reservoir in the riverbed powerhouse. The hydropower produced by the riverbed turbines alone is considered for modelling. The model considers variations in the head over the turbines with the reservoir water level. Also it produces hydropower only within a range of reservoir water levels as may be specified. Irrigation demands into both canals are lumped at the canal head, against which water is withdrawn (irrigation release) from storage. The irrigation release (into the canals) is considered to be a random variable along with reservoir inflow.

The model maximizes the hydropower production from the reservoir with a specified reliability of meeting irrigation demand, using chance constrained linear programming (CCLP). Figure 7.13 gives a schematic layout of the reservoir system.

**The Model**

**Release Policy** The reservoir release policy is defined by a chance constraint. The probability of irrigation release, $IR_t$, in time $t$ equalling or exceeding the irrigation demand, $D_t$, is greater than or equal to a specified value, $P$, where $P$ is the reliability level of meeting irrigation demand.

$$\text{Prob} \left[ IR_t \geq D_t \right] \geq P \quad (7.5.1)$$

**Reservoir Water Balance** The reservoir storage continuity is expressed as

$$S_t + I_t - IR_t - R_t - EV_t = S_{t+1} \quad (7.5.2)$$

where $S_t$ is the total storage at the beginning of period $t$, $I_t$ is the random inflow into the reservoir, $IR_t$ is the total irrigation release, $R_t$ is the downstream release, assumed deterministic, for bed power production, and $EV_t$ is the evaporation loss during period $t$.

$EV_t$ is approximated by a linear relationship,

$$EV_t = \alpha_t + \beta_t (S_t + S_{t+1}) \quad (7.5.3)$$

where $\alpha_t$ and $\beta_t$ are coefficients depending on the period $t$.

Substituting for evaporation term from Eq. 7.5.3 into Eq. 7.5.2, and rearranging

$$IR_t = (1 - \beta_t)S_t - (1 + \beta_t)S_{t+1} + I_t - R_t - \alpha_t \quad (7.5.4)$$

Substituting Eq. (7.5.4) into chance constraint Eq. (7.5.1), one gets

$$\text{Pr} \left[ (1 + \beta_t)S_{t+1} - (1 - \beta_t)S_t + R_t + \alpha_t + D_t \leq I_t \right] \geq P \quad (7.5.5)$$
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The deterministic equivalent is written using the linear decision rule (LDR)

\[ IR_t = S_t + I_t - R_t - EV_t - b_t \]  

(7.5.6)

where \( b_t \) is a deterministic parameter.

As a consequence,

\[ S_{t+1} = b_t \]  

(7.5.7)

Effectively, storage is made deterministic. \( EV_t \) and \( R_t \), both being functions of storage, are deterministic.

### Deterministic Equivalent

The deterministic equivalent of the chance constraint Eq. 7.5.5 is

\[
(1 + \beta b_t - (1 - \beta)b_{t-1} + R_t + c_t + D_t \leq I_t^{1-P})
\]

(7.5.8)

where \( I_t^{1-P} \) is the reservoir inflow during period \( t \), with probability \( 1 - P \), or exceedance probability \( P \). 

### Other Constraints

Storage capacity constraints: The storage in any time period \( t \) should not be less than dead storage capacity \( (K_d) \) and should not exceed the total capacity (including dead storage), \( K_T \).

\[
b_{t+1} \geq K_d \]

(7.5.9)

\[
b_{t-1} \leq K_T
\]

(7.5.10)

with \( b_0 = b_{12} \) for a steady state solution.

Power plant capacity: The energy produced by the riverbed turbine in any time period \( t \), \( EB_t \), should not exceed that corresponding to the installed capacity of the turbine, \( BC \), thus

\[ EB_t \leq BC \]  

(7.5.11)

### Head–Storage Relationship

The reservoir water level and the storage are assumed to be linearly related. The reservoir elevation, \( H_t \), in any period, \( t \), for this purpose is computed as the average of the elevations at the beginning and at end of the period. The following linear relationship is assumed within the range of storages defined by Eqs. 7.5.9 and 7.5.10.

\[
H_t = \gamma[(b_{t+1} + b_{t-1}) + \delta]
\]

(7.5.12)

where \( \gamma \) is the slope of the linear portion of the elevation-storage curve, and \( \delta \) is the intercept.

The net head acting on the turbine is \( H_t - B_{TAIL} \), where \( B_{TAIL} \) is the tail water level.

### Linear Approximation for Power Production Function

A linear approximation of the nonlinear power production term following Loucks et al. (1981) is used. The nonlinear power production term, \( Q \cdot H \), where \( Q \) is the flow and \( H \) is the head, for example, may be linearized by the approximation

\[
Q_t H_t = Q_t \bar{H} + Q_{\bar{H}} H_t - Q_{\bar{H}} \bar{H}
\]

where \( Q_t \) and \( H_t \) are the average values of \( Q \) and \( H \) respectively.

Following this,

\[
EB_t = c \cdot [R_t (H_t - B_{TAIL})]
\]

(7.5.13)
where $R_0$ is the approximate value for the bed power release, $H_t$ is the approximate value for the reservoir elevation, $H_t$ in period $t$, $R_{TAIL}$ is the tail-water elevation of the bed turbine, and $c$ is a constant to convert the product of the rate of flow and the head over the turbine into hydropower produced from the turbine.

The operating range of the reservoir elevation for power production is specified as $H_{min} \leq H_t \leq H_{max}$ for bed turbine operation, $H_{min}$ and $H_{max}$ being specified.

**Objective Function** The objective is to maximize the annual hydropower production by the bed turbine.

$$\text{Maximize } \sum \Delta E_t$$  (7.5.14)

**Methodology** The CCLP model is run for a specified value of $P$ (reliability). Initially, the solution is obtained by assuming some reasonable values $H_0$ and $R_0$, for each $t$. If the values of $H_t$ and $R_t$ in the solution are different from these, then another run is made, replacing $H_0$ and $R_0$ by $H_t$ and $R_t$, respectively. Thus the CCLP model is run successively each time, replacing the values of $R_0$ by $R_t$ and $H_0$ by $H_t$, till convergence is reached, i.e. $H_t = H_0$ and $R_t = R_0$, within a specified tolerance limit.

The model is run for increasing values of $P$, until the solution becomes infeasible. This gives the maximum reliability possible for the given inflow data.

The model is applied to Bhadra Reservoir in Karnataka. The total storage capacity of the reservoir is 2024 Mm$^3$, with a dead storage capacity of 240 Mm$^3$. The installed capacity of the bed turbine is 24,000 kW. Inflow data of 52 years are used in the study. Each time a run is made for a particular value of $P$, the corresponding inflow sequence for the 12 months has to be used.

Table 7.9, for example, gives monthly inflows with $P = 0.65$, along with irrigation demands.

<table>
<thead>
<tr>
<th>Month</th>
<th>Inflow (Mm$^3$)</th>
<th>Irrigation Demand (Mm$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jun</td>
<td>163.40</td>
<td>119.90</td>
</tr>
<tr>
<td>Jul</td>
<td>813.20</td>
<td>136.80</td>
</tr>
<tr>
<td>Aug</td>
<td>702.97</td>
<td>200.60</td>
</tr>
<tr>
<td>Sep</td>
<td>261.73</td>
<td>195.80</td>
</tr>
<tr>
<td>Oct</td>
<td>302.81</td>
<td>203.20</td>
</tr>
<tr>
<td>Nov</td>
<td>89.31</td>
<td>189.70</td>
</tr>
<tr>
<td>Dec</td>
<td>50.52</td>
<td>109.40</td>
</tr>
<tr>
<td>Jan</td>
<td>26.93</td>
<td>137.30</td>
</tr>
<tr>
<td>Feb</td>
<td>17.10</td>
<td>180.10</td>
</tr>
<tr>
<td>Mar</td>
<td>10.64</td>
<td>197.90</td>
</tr>
<tr>
<td>Apr</td>
<td>11.70</td>
<td>197.90</td>
</tr>
<tr>
<td>May</td>
<td>11.06</td>
<td>178.60</td>
</tr>
</tbody>
</table>
The model solution shows that the maximum possible reliability of meeting irrigation demands is 0.65, or 65%, and the corresponding maximum annual hydro-energy produced by the bed turbine is 5.68 M kwh.

Figure 7.14 shows a plot of reliability versus annual hydro-energy produced from the riverbed turbine. The curve gives the energy that can be produced at other levels of reliability.

From the curve, one can find the maximum annual energy that can be produced by the bed turbine for a specified reliability of meeting irrigation demands; or, alternatively, one may find the reliability of meeting irrigation demands at a given level of hydropower production by the riverbed turbine.

REFERENCES


**Further Reading**

In this chapter we discuss some applications of dynamic programming in water resources, with a focus on irrigation water allocation.

**8.1 OPTIMAL CROP WATER ALLOCATION**

We first consider the problem of allocating a known amount of water in an intraseason period among crops in that period. This problem may be formulated as a dynamic programming (DP) problem, using the backward recursion of DP. Each crop constitutes a stage in the DP. The state variable is the amount of water available \( w \) at a given stage for allocation among all the stages up to and including that stage.

Let \( g_r(w) \) be the minimum value of the objective function (a measure of evapotranspiration deficit) when \( w \) is allocated to \( r \) stages up to and including that stage, and \( w_r \) be the allocation made at \( r \)th stage to the crop corresponding to that stage. The allocation \( w_r \), which is in volume units, is divided by the area of the crop \( A_c \) for which it is allocated to get \( x_r \), the irrigation depth applied to the crop, which is used to determine the available soil moisture, \( \theta_{ac} \), and the actual evapotranspiration, \( ET_{ac} \).

At stage 1, i.e. with only the last crop \( NC \) considered for allocation, \( (r = 1) \)
\[
g_1(w) = \min \left[ \Phi_{NC} \right]
\]
\[0 \leq w_1 \leq w \leq X_t\]

where \( X_t \) is the known amount of water available for allocation, \( NC \) is the last crop, \( \Phi_{NC} \) is a measure of deficit allocation (taken usually as the value of evapotranspiration deficit) and \( w_1 \) is the possible water allocation for stage 1.

At stage 2, when the last two crops remain to be allocated \( (r = 2) \)
\[
g_2(w) = \min \left[ \Phi_{NC-1} + g_1(w - w_2) \right]
\]
\[0 \leq w_2 \leq w \leq X_t\]
Proceeding this way, the recursive relation for the \( r \)th stage may be expressed as

\[
g_r(w) = \min_{w_r, \leq w} \{ \Phi_{NC,r+1} + g_{r+1}(w - w_r) \}
\]

\[0 \leq w_r \leq w \leq X_r\]

At the last stage, \( r = NC \), all the crops remain to be allocated, and \( w = X_r \).

\[
g_{NC}(w = X_r) = \min_{w_r} \{ \Phi'_r + g_{NC-1}(w - w_{NC}) \}
\]

\( g_{NC}(X_r) \) represents the minimum value of the objective function for optimal allocation of the available water, \( X_r \), among all the \( NC \) number of crops in the time period \( t \).

Once the allocation model is solved, the optimal allocation for each crop is known. The optimal allocation, which is in volume units, is divided by the area of crop \( c \), to get \( x_t^c \) in depth units. With these optimal allocations, the soil moisture balance is carried out to determine the soil moisture of crops, \( \{n_1, n_2, \ldots, n_{NC}\} \) at the beginning of the next time period \( t + 1 \) (see Fig. 8.1). This defines one value of the soil moisture at the beginning of the period \( t + 1 \) for each crop \( c \). The initial soil moisture vector for the period \( t + 1 \) over all the crops is \( \mathbf{N} = \{n_1, n_2, \ldots, n_{NC}\} \).

This allocation model is used in deriving the steady state operating policy (Sec. 8.2) and the real-time operating policy (Sec. 8.3).

### 8.2 STEADY STATE RESERVOIR OPERATING POLICY FOR IRRIGATION

Optimization models for long-term reservoir operation, integrating the reservoir release decisions with the irrigation allocation decisions at the field level, are formulated to conceptually operate in two phases. In the first phase, deterministic dynamic programming is used to allocate a known amount of water among all crops to optimize the impact of allocation in a period (with impact
measured with respect to the evapotranspiration deficit and sensitivity of the crop yield to a deficit supply in that period. Such allocations are determined for all possible supplies in a given period, for all periods in a year. In the second phase, a stochastic dynamic programming model evaluates all the intraseasonal periods to optimize the overall impact of the allocation over a full year. The end result of this two-phase analysis is a set of decisions indicating the reservoir releases to be made in each intraseason period and the distribution among the crops of this release, available at the crop level after accounting for losses between the reservoir and the application area. As the policy specifies a steady state operation, the cropping pattern used in the SDP model is assumed to remain fixed over the entire operating horizon.

Figure 8.2 shows the state transformation for the SDP problem for reservoir operation for irrigation, using backward recursion. The four-state variables being considered are: reservoir storage, inflow, soil moisture, and rainfall in the command area. The recursive relationship for this four-state variable SDP model is written as,

\[ f_t^1(k, i, m, p) = \max \{ G(k, i, l, m, p, t) + \sum_{q} P^{s+1}(q) P^q f_{t+1}^2(l, j, n, q) \} \]

where,

- \( f_t^1(k, i, m, p) \) is the optimal value of the objective function in period \( t \) (stage \( s \)), with reservoir storage class \( k \), inflow class \( i \), soil moisture class \( m \), and rainfall class \( p \).
- \( G(k, i, l, m, p, t) \) is a measure of the system performance, corresponding to the discrete classes \( k, i, l, m, \) and \( p \) in period \( t \), and is obtained by solving the allocation problem.
\( P_{t+1}(q) \) is the probability of rainfall in class \( q \) in period \( t + 1 \), and \( P_{t+1}^j \) is the transition probability of inflow being in class \( j \) in period \( t + 1 \) given that it is in class \( i \) in period \( t \).

The system performance measure, \( G(k, i, l, m, p, t) \), is obtained for a given \( k, l, m, \) and \( p \) in period \( t \) by solving a one-dimensional dynamic programming (DP) problem for allocation of a known amount of water among crops with known soil moistures in period \( t \). The optimal objective function of the DP problem is defined as \( G(k, i, l, m, p, t) \).

**Steady State Solution**

When Eq. (8.2.1) is solved recursively, a steady state solution is reached fairly quickly (within about 4 to 5 cycles). The decision variable in the optimization is the end-of-the-period storage class \( l \), in a period \( t \) for given values of \( k, i, m, \) and \( p \). The steady state operating policy is specified as the optimal end-of-the-period storage class, \( P^* \), to be maintained for a given initial storage, inflow, rainfall, and soil moisture of each crop. The optimal release from the reservoir is obtained from the storage continuity corresponding to the storage class interval \( P^* \). The crop water allocations out of this release are specified through the allocation model, after accounting for conveyance and other losses. The SDP model falls prey to the ‘curse of dimensionality’, as the number of state variables increases. For a case study of the Malaprabha Reservoir in South India the SDP model expressed in (8.1), bordered on computational intractability, with 15 class intervals for storage, four for inflow, five for soil moisture, one for each crop (5 crops) and four class intervals for rainfall and with 36 within-year time periods. The model therefore remains restricted to a single purpose, single-reservoir model, and an extension to include multiple reservoirs or multiple purposes is possible only by sacrificing a great deal of crop-level details (e.g. by excluding soil moisture and rainfall state variables) in the model. We present here the summary results of the long-term steady state policy model, when applied to the Malaprabha reservoir in South India. The reservoir has a live storage capacity of 870 Mm\(^3\). The crop details are given in Table 8.1.

Long-term policies obtained for two typical periods, one in the kharif season (period 6) and the other in the rabi season (period 22) are shown in Figs. 8.3 and 8.4. Note that the state discretization scheme used is different for different seasons. For example, \( k = 2 \) in two different periods, in general, corresponds to two different initial storages. As the variation in rainfall in the command area is not high, inclusion of the stochastic state variable does not significantly change the policy, as may be seen from the two figures. The implications of the long-term policies must be studied separately through simulation (e.g. Mujumdar and Vedula, 1992).
Table 8.1 | Crops, Cropped Areas and Crop Duration

<table>
<thead>
<tr>
<th>Crop</th>
<th>Cropped Area, ha</th>
<th>Duration, t</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Kharif Season</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Maize</td>
<td>33784</td>
<td>1-12</td>
</tr>
<tr>
<td>Pulses</td>
<td>16892</td>
<td>1-11</td>
</tr>
<tr>
<td>Sorghum</td>
<td>84462</td>
<td>1-12</td>
</tr>
<tr>
<td><strong>Rabi Season</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sorghum</td>
<td>33784</td>
<td>16-27</td>
</tr>
<tr>
<td>Pulses</td>
<td>16892</td>
<td>16-26</td>
</tr>
<tr>
<td>Safflower</td>
<td>16892</td>
<td>16-27</td>
</tr>
<tr>
<td>Wheat</td>
<td>67570</td>
<td>16-25</td>
</tr>
<tr>
<td><strong>Two Seasonal Crop</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cotton</td>
<td>67570</td>
<td>7-31</td>
</tr>
</tbody>
</table>

Fig. 8.3 | Steady State Policy for a Kharif Period (Period # 6)

Fig. 8.4 | Steady State Policy for a Rabi Period (Period # 22)
The problem considered in this section is one of making reservoir release and crop water allocation decisions in real time, given the current state of the system. The state of the system is defined by the reservoir storage, crop soil moisture, and the current growth status of the crops, obtained as a measure of evapotranspiration deficits until the current period in real time. Intraseason periods, typically consisting of a week to ten days, are used in the model to provide adaptive real-time decisions.

Dynamic programming is used for arriving at adaptive real-time operating policies for an irrigation reservoir system. The real-time operation model may be formulated as a dynamic programming problem, to be solved once at the beginning of each intraseasonal period. It uses forecasted inflows for the current period in real time (for which a decision is sought) and all subsequent periods in the year. These forecasts themselves are obtained using the latest available information on the previous period's inflow. Solving the real-time operation model gives the release decisions, and the crop water allocations for all periods in the year starting from the current period. Only the decisions on release and allocations for the current period are implemented and the state of the system (comprising the reservoir storage, soil moisture of each crop, and a crop production measure indicating the current state of the crop) is updated at the end of the period. Because the model is applied in real-time, the productive value of previous allocations to a crop, up to the beginning of the current period in real time would be known. This information is taken as an input to the model in deriving optimal releases for the subsequent periods in the year.

The model thus updates the release decisions from period to period, making use of the latest available information. Actual rainfall in the command area, reservoir inflow, current production status (estimated from the evapotranspiration deficits from the beginning of the crop season until the current time period in real time), and actual soil moistures of the crops, all contribute to the updating of the release decisions for the subsequent periods.

The irrigation allocation to a crop in a period is based on
1. its current production status, which is the net effect of water supplied to the crop (through irrigation allocations and precipitation) from the beginning of the season up to the beginning of that period,
2. available soil moisture in the root zone of the crop, and
3. competition for water with other crops.

The first two conditions are introduced in the mathematical model through the use of two-state variables, a crop production state variable and a soil moisture state variable for each crop. The third condition of competition with other crops is introduced through use of crop yield factors, $K_y$, in the objective function, which indicate the sensitivity of a crop to a deficit supply, and which vary with the crop growth stages. The state variable for crop production indicates the production potential of a crop from the current period to the end of the crop
Applications of Dynamic Programming

Features of the real-time operation model are shown in Fig. 8.5. In this figure, \( t_p \) is the current period in real time, \( T \) is the last period in the year, and \( I_t \) is the inflow during period \( t \). The operating policy model, which scans across time periods from the current period to the last period in the year is formulated as a deterministic dynamic programming model. The allocation model, which provides decisions on crop water allocations in an intraseasonal period, is solved for a given amount of available water, known soil moisture of each crop, and a given crop production measure of each crop.

The recursive relationship for the operation model is written, for any intermediate period \( t \), as:

\[
f_t^j(k, \text{M}, \text{M}_b) = \max \{ \phi(k, l, \text{M}, \text{M}_b, t) + f_{t+1}^{j-1}(l, \text{N}, \text{M}_e) \} \tag{8.3.1}
\]

where,

- \( \text{M} \) is the soil moisture vector at the beginning of period, \( t \)
- \( \text{M}_b \) is the crop production measure vector at the beginning of period, \( t \)
- \( \text{N} \) is the soil moisture vector at the end of period, \( t \)
- \( \text{M}_e \) is the crop production measure vector at the end of period, \( t \)
- \( \phi(k, l, \text{M}, \text{M}_b, t) \) is the system performance measure at period \( t \), and,
- \( f_t^j(k, \text{M}, \text{M}_b) \) is the optimal accumulated system performance measure up to period \( t \), stage \( j \).

The crop production measure vector, \( \text{M}_b \), is estimated in real time from the allocations already made to the crop in the season until the current period, based on the evapotranspiration deficits. The production measure, \( \text{M}_e \), results from the allocations being considered in the current period. The system performance measure, \( \phi(k, l, \text{M}, \text{M}_b, t) \), is obtained from the solution of the
allocation model. The real-time operation model needs to be solved once at the beginning of each period in real time, specifying the current state of the system, in terms of the crop production measure, crop soil moisture, and reservoir storage. The crop production measure may be computed from production functions, based on the actual allocations made to the crop up to the beginning of the current period in real time. Solution of the real time operation model specifies the reservoir release for all subsequent periods, including the current period, along with crop water allocations. An application of this model for a real case study may be seen in Mujumdar and Ramesh (1997).

A major difference between the LP model presented for short-term operation in Section 7.4.1 and the DP model presented here is that while in the DP model two dynamic programming formulations are used—one for obtaining crop water allocations and the other for deriving the operating policy—in the LP model, a single integrated LP formulation is used to serve the same problem of specifying reservoir release and crop water allocations. The major limitations of the DP model are

(a) A large number of state variables are needed for its application in real situations, making it computationally intractable in many cases, and
(b) The crop water allocation model, being solved externally during each intraseasonal period for a given set of state variables in the operation policy model, restricts the allocations to a set of discrete values only.

Both these limitations are overcome in the LP model by replacing the two DP formulations by a single LP formulation.

REFERENCES


Further Reading


In this chapter, a brief description of the Artificial Intelligence (AI) tools, with their applications for water resources systems problems is provided. We begin with a discussion of the Artificial Neural Networks (ANN) and then discuss a few applications of fuzzy logic and fuzzy optimization.

9.1 ARTIFICIAL NEURAL NETWORKS

9.1.1 Basic Principles

Artificial Neural Network (ANN) is a relatively new technique used in hydrologic and water resources systems modelling. An ANN is structured to resemble the biological neural network in two aspects:

(i) Knowledge acquisition through a learning process, and
(ii) Storage of knowledge through connections, known as synaptic weights.

ANNs are particularly useful as pattern-recognition tools for generalization of input–output relationships. The most common applications of ANNs in water resources include those for rainfall runoff relationships and streamflow forecasting.

In this section a brief introduction to the artificial neural network is provided and an application to inflow forecasting is discussed briefly. For a more rigorous understanding of ANNs, the reader is referred to Bishop (1995) and Hykin (1994). An excellent review of ANN applications in hydrology is available in ASCE (2000 a, b).

An ANN consists of interconnected neurons receiving one or more inputs and resulting in an output. Figure 9.1 shows one such neuron.

The input to a simple ANN model for a rainfall–runoff relationship may be the rainfall during a period, and the output from the ANN will be the corresponding runoff. An ANN is trained to learn the relationship between the input and the output. To achieve the training, a number of sets of inputs and outputs are presented to the ANN. A training algorithm achieves the training by determining the optimal weights.
Recent Modelling Tools

Each input link \( i (i = 1, 2, 3) \) has an associated external input signal or stimulus \( x_i \) and a corresponding weight \( w_i \) (Fig. 9.1). A schematic black box equivalent of the neuron is illustrated in Fig. 9.2. The neuron behaves as an activation or mapping function \( f(\cdot) \), producing an output \( y = f(z) \), where \( z \) is the cumulative input stimuli to the neuron and \( f \) is typically a nonlinear function of \( z \). The neuron calculates the weighted sum of its inputs as

\[
z = w_1 x_1 + w_2 x_2 + w_3 x_3
\]

The function \( f(z) \) is typically a monotonic nondecreasing function of \( z \). Some examples of the commonly used activation functions \( f(z) \) are shown in Fig. 9.3.

Back propagation (BP) and Radian Basis Function (RBF) are the most commonly used network-training algorithms used in hydrologic and water resources applications, and are discussed here.
Back Propagation
Back propagation is perhaps the most popular algorithm for training ANNs. The network architecture used in a back-propagation (BP) algorithm is made up of interconnected nodes arranged in at least three layers (Fig. 9.4). The input layer receives the input data patterns. The output layer produces the result. The hidden layers sequentially transform the input into the output. A maximum of three hidden layers have been found to be adequate for most problems in water resources applications. The number of neurons in the hidden layers is usually fixed by trial and error.

The transfer function used in BP networks is usually a sigmoid function (Fig. 9.3). During training, back-propagation networks process the patterns in a two-step procedure. In the first or forward phase of back-propagation learning, an input pattern is applied to the network, with initially assumed weights to provide the output at the output layer. The error is estimated from the corresponding output value provided in the training set. In the second or backward phase, the error from the output is propagated back through the network to adjust the interconnection weights between layers. This process is repeated until the network's output is acceptable. Back-propagation learning is this process of adapting the connection weights. When presented with an unknown input pattern the trained network produces a corresponding output pattern. Details of the back-propagation method are described by Freeman and Skapura (1991).

Radial Basis Function Network
Radial basis neural networks (Fig. 9.5) are feed-forward neural networks with only one hidden layer and radial basis functions (RBF) as activation functions. The hidden layer performs a fixed nonlinear transformation with no adjustable parameters. The radial basis...
function networks are trained with a modified form of gradient descent training. The training algorithm of RBF networks may be found in Leonard et al. (1992) and Haykin (1994). The primary difference between the RBF network and back propagation is in the nature of the nonlinearities associated with the hidden node. The nonlinearity in back propagation is implemented by a fixed function such as a sigmoid. The RBF method, on the other hand, bases its nonlinearities on the data in the training set (ASCE, 2000a). For most problems, the back-propagation algorithm requires higher training times than the RBF networks. With this advantage, the RBF networks are ideally suited for applications involving real-time forecasting.

9.1.2 Advantages and Limitations of ANNs

Some advantages and limitations of ANNs discussed by Nagesh Kumar (2003) are listed here.

1. **Nonlinearity** Nonlinearity is a useful property of neural networks, particularly if the underlying physical mechanism responsible for generation of the output is inherently nonlinear (e.g., runoff from a watershed).

2. **Input–Output Mapping** A popular paradigm of learning called supervised learning involves modification of the synaptic weights of a neural network by applying a set of labelled training samples or task examples. Thus the network learns from the examples by constructing an input–output mapping for the problem at hand.

3. **Adaptivity** A neural network trained to operate in a specific environment can be easily retrained to deal with minor changes in the operating environmental conditions. Moreover, when it is operating in a nonstationary environment (i.e. where statistics change with time), a neural network can be designed to change its synaptic weights in real time (e.g. modelling nonstationary hydrologic time series).
4. **Evidential Response** In the context of pattern classification, a neural network can be designed to provide information not only about which particular pattern to select, but also about the confidence in the decision made. This latter information may be used to reject ambiguous patterns, and thereby improve the classification performance of the network.

A major limitation of ANNs arises due to their inability to learn the underlying physical processes. Another difficulty in using ANNs is that there is no standardized way of selecting the network architecture. The choice of network architecture, training algorithm, and definition of error are usually determined by the user’s experience and preference, rather than the physical features of the problem. Very often, ANN training is performed with limited length of hydrologic data. Under these circumstances, it will be difficult to say when the generalization will fail, to decide the range of applicability of the ANN. In addition, the following limitations of ANNs must be kept in view.

5. **Long Training Process** For complex problems, the direction of error reduction is not very clear and may lead to some confusion regarding the continuation of the training process. The training process can be very long and sometimes futile for a bad initial weight space.

6. **Moving Target** The problem of a moving target arises as the weights have to continuously adjust their values from one output to another for successive patterns. The change in a weight during one training pass may be nullified in the next pass because of a different training pattern.

### 9.1.3 An Example Application for Inflow Forecasting

A simple application of ANN for one-time step-ahead inflow forecasting is discussed in this section. The training algorithm available in the MATLAB package is used for training the RBF network. The inputs presented to the neural network models are the ten-day inflow to a reservoir in a period and the time period itself. The ten-day inflow of the next period forms the output (see Fig. 9.6). Sets of input–output pairs are used for training. Using the time period as one of the inputs helps the network to learn the relationship between the time period and the inflow values.

![Fig. 9.6 Structure of the Radial Basis Function Network for the Example Application](image-url)
Training comprises presentation of input and output pairs to the network and obtaining the interconnection weights. The network is presented with a sufficiently large number of input and output pairs, repeatedly, till the error, summed over all the training pairs, is below an acceptable value set \textit{a priori}. This value, called the system error, may be set to a reasonably low value to give the desired network performance. Using the interconnection weights and the transfer function for the hidden and output layers, the output value for the known input values are calculated.

Table 9.1 gives the network parameter values used for the RBF training. Once the training is complete, the network stores the values of interconnection weights between hidden layer and output layer and the centers of Radial Basis Function units. Actual inflows for a period of 39 years in a South Indian river are used for this application. Thirty-six years of the data are used for the training and the rest for testing.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input layer neurons</td>
<td>2</td>
</tr>
<tr>
<td>Hidden layer neurons</td>
<td>50</td>
</tr>
<tr>
<td>Output layer neurons</td>
<td>1</td>
</tr>
<tr>
<td>Spread</td>
<td>0.018</td>
</tr>
<tr>
<td>System error</td>
<td>1</td>
</tr>
</tbody>
</table>

* required in the MATLAB Neural Network tool box.

The performance of the model assessed using square error is obtained by

\[
E(p) = \frac{1}{2} \left[ A(p) - O(p) \right]^2
\]

where \( E(p) \) is the square error for ten-day period \( p \), \( A(p) \) is the actual observed inflow and \( O(p) \) is the forecasted inflow for the time period \( p \). In this case, the trained RBF network yields a square error of 0.014523 over the testing period of three years. Acceptability of this value of error will depend on whether a better model (in terms of the resulting forecasting error) may be developed for the same purpose or not. Figure 9.7 shows a comparison of the historical inflows with the forecasted inflows in the testing period of three years. It may be noted that, in this particular case, while the ANN performs reasonably well in reproducing normal flows, its performance is poor in case of peak flows. Note also that the neural network generates a negative forecast of the flow.

Note also that the manner in which the data is normalized, and, in general, by choosing an appropriate normalization of the training data, negative values in the output may be avoided.

### 9.2 FUZZY SETS AND FUZZY LOGIC

Uncertainties in water resources systems arise not only due to randomness of hydrologic variables but also due to imprecision, subjectivity, vagueness associated with decision making and lack of adequate data. Such uncertainties are
best addressed through fuzzy logic. We will briefly review the mathematical concepts of fuzzy logic in this section and discuss its application to reservoir operation modelling.

9.2.1 Introduction

Some Basic Concepts of Fuzzy Logic

Some basic concepts of the fuzzy logic and operations are discussed briefly in this section. For a more detailed coverage refer Klir & Folger (1995), Zimmermann (1996), Kosko (1996), and Ross (1997).

Membership Functions

A membership function (MF) is a function—normally represented by a geometric shape—that defines how each point in the input space is mapped to a membership value between 0 and 1. If \( X \) is the input space (e.g. inflow to a reservoir) and its elements are denoted by \( x \), then a fuzzy set \( A \) in \( X \) is defined as a set of ordered pairs

\[ A = \{ x, \mu_A(x)/x \in X \} \]

where \( \mu_A(x) \) is called the membership function of \( x \) in \( A \). Thus the membership function maps each element of \( X \) to a membership value between 0 and 1. A membership function can be of any valid geometric shape. Some commonly used membership functions are of triangular, trapezoidal, and bell shape.

An important step in applying methods of fuzzy logic is the assessment of the membership function of a variable. For reservoir operation modelling purposes, the membership functions required are typically those of inflow, storage, demand, and release. When the standard deviation of a variable is not large, it is appropriate to use a simple membership function consisting of only straight lines, such as a triangular or a trapezoidal membership function.
Kosko (1996) observed that the fuzzy controller attained its best performance when there is an overlapping in the adjacent membership functions. A good rule of thumb is that the adjacent membership functions should overlap approximately 25 per cent.

**Fuzzy Rules**

A fuzzy rule system is defined as the set of rules which consists of sets of input variables or premises $A$, in the form of fuzzy sets with membership functions $\mu_A$, and a set of consequences $B$ also in the form of a fuzzy set. Typically a fuzzy if-then rule assumes the form

$$\text{if } x \text{ is } A \text{ then } y \text{ is } B$$

where $A$ and $B$ are linguistic values defined by fuzzy sets on the variables $X$ and $Y$, respectively. The ‘if’ part of the rule ‘$x$ is $A$’ is called antecedent or premise, and the ‘then’ part of the rule ‘$y$ is $B$’ is called consequence. In the case of binary or two-valued logic, if the premise is true then the consequence is also true. In a fuzzy rule, if the premise is true to some degree of membership, then the consequence is also true to that same degree.

The premise and consequence of a rule can have several parts like

$$\text{if } x \text{ is } A \text{ and } y \text{ is } B \text{ and } z \text{ is } C, \text{ then } m \text{ is } N \text{ and } o \text{ is } P, \text{ etc.}$$

**9.2.2 Fuzzy Rule-based Reservoir Operation Model**

The fuzzy logic-based modelling of a reservoir operation is a simple approach, which operates on an ‘if-then’ principle, where ‘if’ is a vector of fuzzy explanatory variables or premises such as the present reservoir pool elevation, the inflow, the demand, and time of the year. The ‘then’ is a fuzzy consequence such as release from the reservoir.

In modelling of reservoir operation with fuzzy logic, the following distinct steps are followed:

1. Fuzzification of inputs, where the crisp inputs such as the inflow, reservoir storage, and release are transformed into fuzzy variables,
2. Formulation of the fuzzy rule set, based on an expert knowledge base,
3. Application of a fuzzy operator, to obtain one number representing the premise of each rule,
4. Shaping of the consequence of the rule by implication, and
5. Defuzzification.

These steps are discussed in the following paragraphs for a general reservoir operation problem.

**Fuzzification of the Inputs**

The first step in building a fuzzy inference system is to determine the degree to which the inputs belong to each of the appropriate fuzzy sets through the membership functions. The problem of constructing a membership function is that of capturing the meaning of the linguistic terms employed in a particular application adequately and of assigning the meanings of associated operations to the linguistic terms. Useful discussions related to construction of membership functions may be found in Dombi (1990) and Klir & Yuan (1997).
Figure 9.8 shows the transformation of the storage variable through its membership function as an example. This membership function may indicate the degree to which a particular storage value belongs to the set of ‘high’ storages. A storage of 200 units is not high, in this case, and therefore has a membership value of 0. On the other hand, the storage of 500 units may be classified as ‘very high’, and its membership value is also zero in the membership function of ‘high’ storage. Other storage values have membership function values between 0 and 1, according to the degree to which they are ‘high’. For a storage of 300 Mm$^3$, for example, the membership value is 0.7.

### Formulation of the Fuzzy Rule Set
The fuzzy rule set is formulated based on expert knowledge. A fuzzy rule may be of the form: If the storage is low and the inflow is medium in period $t$, then the release is low. The expert knowledge available on the particular reservoir is used for formulating the rule base.

### Application of Fuzzy Operator
Once the inputs are fuzzified, the degree to which each part of the premise has been satisfied for each rule is known. If the premise of a given rule has more than one part, then a fuzzy operator is applied to obtain one number that represents the result of the premises for that rule. The input to the fuzzy operator may be from two or more membership functions, but the output is a single value.

The fuzzy logic operators such as the AND or OR operators obey the classical two valued logic. The AND operator can be conjunction (min) of the classical logic or it can be the product (prod) of the two parameters involved in it. Similarly, the OR method can be the disjunction operation (max) in the classical logic or it can be the probabilistic OR (probor) method. The probabilistic OR method is defined by the equation
\[
\text{probor}(m, n) = m + n - mn
\]

The result of the ‘AND’ (min) operator for the example shown in Fig. 9.9 will be 0.3.
For example, consider the input 1 as storage and the input 2 as inflow. Let the premise of the rule be of the nature

\[
\text{if storage is } x \text{ or inflow is } y \text{ then } \ldots
\]

The two different parts of the premise yield the membership values of 0.3 and 0.8 for storage and inflow respectively. The fuzzy operator is applied to obtain one number that represents the result of the premise for that rule; in this case, the fuzzy \textit{OR} operator simply selects the maximum of the two values, 0.8, and the fuzzy operation for the rule is complete. If the rule uses the fuzzy probabilistic \textit{OR} (probor) operator then the result will be 0.86.

### Implication

The fuzzy operator operates on the input fuzzy sets to provide a single value corresponding to the inputs in the premise. The next step is to apply this result on the output membership function to obtain a fuzzy set for the rule. This is done by the implication method. The input for the implication method is a single number resulting from the premise, and the result of implication is a fuzzy set. Implication occurs for each rule by the \textit{AND} method that truncates the output fuzzy set, or the \textit{prod} method that scales the output fuzzy set.

The implication method is shown in Fig. 9.10. In the figure, the truncation of the output fuzzy set is done at the lower of the two membership function values, resulting from the input2, because the \textit{AND} fuzzy operator is used on the inputs.

### Aggregation

Aggregation is the unification of the output of each rule by merely joining them. When an input value belongs to the intersection of the two membership functions, fuzzy rules corresponding to both the membership functions are invoked. Each of these rules, after implication, specifies one-output fuzzy set. The two-output fuzzy sets are then unified to get one single-output fuzzy set. Aggregation is shown in Fig. 9.11. If more than one input lies in the intersection regions, all corresponding rules are invoked and aggregation is carried out on the output fuzzy sets. The input to the aggregation process is the truncated output fuzzy sets obtained from the implication process of each rule. The output of the aggregation process is one fuzzy set.
Defuzzification  The result obtained from implication and aggregation, if necessary, is in the form of a fuzzy set. For application, this is defuzzified. The input for the defuzzification process is a fuzzy set and the result is a single crisp number. The most common defuzzification method is the “centroid” evaluation, which simply chooses the centre of area defined by the fuzzy set.

An Example Application
In this example application, fuzzy rules are generated from the simulated operation of a reservoir with a steady state policy derived from an SDP model.
Reservoir operation is simulated for 30 years, with 36 within-year periods. The simulation results thus contain 30 values of reservoir storage, inflow, and release for each of the 36 within-year periods. These values are matched to the respective fuzzy sets (e.g. low, medium, etc.) based on the interval to which they belong. The database so generated will thus have information such as ‘Storage: Low; Inflow: Medium; Release: Low’ and so on for each of the 36 periods for the 30 years of simulation. This database is used to formulate the fuzzy rules for individual periods. From the database, information is picked up on the consequence (e.g. Release: Low) for each premise (e.g. Storage: Low, Inflow: Very Low, and Time-of-Year: 22). Where one premise leads to more than one consequence (e.g. Release: Medium and Release: High), in the database for the same period in different years, the fuzzy rule for the condition is formulated based on the average value of the simulated release. Suppose, for example, in a particular year in simulation, for period 22, the storage, inflow, and release values from the SDP simulation are 180, 10, and 40, respectively. From the membership function definitions (Fig. 9.12) these values are traced to 'low', 'very low' and 'low-med' of storage, inflow, and release membership functions, respectively. If there are no other years in simulation for which the premise (Storage: Low and Inflow: Very Low) is obtained for period 22, then the rule formulated in this case will be:

If the storage is low and the inflow is very low and time-of-year is period 22 then release is low-med.

If, however, there is more than one year in simulation for which the same premise (Storage: Low and Inflow: Very Low) is obtained for the same period, and the consequences are different, the fuzzy rule is formulated for the premise based on the average value of the consequences. To illustrate this, for the above example, suppose in any two years (out of 30 years of simulation), the storage, inflow, and release are 180, 10, and 40 and 100, 25, and 55 for the two years, respectively. As may be verified from Fig. 9.12, the premise arising out of this data would be ‘Storage: Low; and Inflow: Very Low’ for both the cases, but the consequences are ‘Release: Low-Medium’ for one year and ‘Release: Very-High’ for the other. In such a case, the consequence is formulated based on the average value of the release (i.e. (40 + 55)/2 = 47.5). This value is traced to the membership of both medium and medium-high from Fig. 9.12. As 47.5 has a higher membership value (using the OR operator), for the membership function ‘Medium’, the fuzzy rule is written as

If the storage is low and the inflow is very low and time-of-year is period 22 then release is medium.

The number of rules in the fuzzy rule base is \( \prod_{i=1}^{n} c_i \), where \( c_i \) is the number of classes for the \( i \)th variable and \( n \) is the number of variables. With increasing number of classes for the variables, a greater accuracy may be achieved. However, a very large rule base leads to dimensionality problems.
Application of Fuzzy Operator

The premise part of the rule is assigned one membership value through this operation. For example, in period 22, a storage of 180 corresponds to a membership value of 0.7 in the membership function for ‘low’ storage, and the inflow of 10 corresponds to 1.0 in the membership function for ‘very low’ inflow. The result of application of the fuzzy operator ‘AND’ will be to pick the minimum of these two (viz. 0.7) and assign it to the premise of the rule.

Implication, Aggregation, and Defuzzification

The result obtained from application of the fuzzy operator is applied to the membership function of the consequence of the rule, and one output fuzzy set is obtained. For the example mentioned earlier, the consequence is ‘Release: Medium’. The membership function of this fuzzy set is truncated as shown in Fig. 9.9, at the level of 0.7, which corresponds to the membership value of the premise assigned in the
previous step, and the resulting truncated output fuzzy set is obtained. If any of the input values belong to the intersection region of two membership functions, the rules corresponding to both the membership functions need to be invoked. In such a case the same set of input values will yield different output sets. To obtain one output set corresponding to these different output sets, aggregation of output fuzzy sets (Fig. 9.11) is carried out. The end result of implication and aggregation, if necessary, is a truncated fuzzy set for release which is defuzzified using the centroid method to obtain the crisp value of release.

9.3 Fuzzy Linear Programming

9.3.1 Fuzzy Decision

The concept of fuzzy decision was first introduced by Bellman and Zadeh (1970). The imprecisely defined goals and constraints are represented as fuzzy sets in the space of alternatives. The confluence of fuzzy goals and fuzzy constraints is defined as the fuzzy decision. Considering a fuzzy goal, $F$, and a fuzzy constraint, $C$, the fuzzy decision, $Z$, is defined as the fuzzy set resulting from the intersection of $F$ and $C$. Figure 9.13 shows the concept of a fuzzy decision. Mathematically,

$$Z = F \cap C$$

Fig. 9.13 | Fuzzy Decision

The membership function of the fuzzy decision $Z$ is given by

$$\mu_Z(x) = \min \{ \mu_F(x), \mu_C(x) \}$$

The solution $x^*$, corresponding to the maximum value of the membership function of the resulting decision $Z$, is the optimum solution. That is,

$$\mu_Z(x^*) = \lambda^* = \max_{x \in \mathbb{R}} \mu_Z(x)$$

Goals and constraints are treated identically in fuzzy multiple objective optimization. Representing the fuzzy goals and fuzzy constraints by fuzzy sets $F_i, i = 1, 2, \ldots, n_j$ the resulting decision can be defined as

$$Z = \bigcap_{i=1}^{n_j} F_i$$
In terms of the corresponding membership functions the resulting decision for the multiple objective problem is

$$\mu_l(X) = \min_i \{ \mu_i(X) \}$$

where $X$ is the space of alternatives. The optimal solution $X^*$ is given by

$$\mu_l(X^*) = \hat{\lambda}^* = \max_{X \in Z} \{ \mu_l(X) \}$$

Usually the space of alternatives (i.e. the decision space) is restricted by precisely defined constraints known as crisp constraints (e.g. mass balance of flows at a junction in a river network for a water allocation problem; minimum waste treatment level imposed on the dischargers by the pollution control agency for a waste load allocation problem). Incorporating these crisp constraints, $g_j(X) \leq 0$, $j = 1, 2, \ldots, n_c$, the crisp equivalent of the fuzzy multiple objective optimization problem can be stated as follows (Zimmermann 1978; Kindler 1992; Rao 1993; Sakawa 1995).

$$\max \hat{\lambda}$$

subject to

$$\mu_l(X) \geq \hat{\lambda} \quad \forall \; i$$

$$g_j(X) \leq 0 \quad \forall \; j$$

$$0 \leq \hat{\lambda} \leq 1$$

### 9.3.2 Fuzzy Linear Programming for River Water Quality Management

Water quality management of a river system may be viewed as a multi-objective optimization problem with conflicting goals of those who are responsible for maintaining the water quality of the river system (e.g. pollution control agencies) and those who make use of the assimilative capacity of the river system by discharging the waste to the water body (e.g. industries). The goal of the pollution control agency is to ensure that the pollution is within an acceptable limit by imposing certain water quality and effluent standards. On the other hand, the dischargers prefer to make use of the assimilative capacity of the river system to minimize the waste treatment cost.

Concentration level of the water quality parameter $i$ at the checkpoint $l$ is denoted as $C_{il}$. The pollution control agency sets a desirable level, $C_i^0$, and a minimum permissible level, $C_i^L$, for the water quality parameter $i$ at the checkpoint $l$ ($C_i^L < C_i^0$).

### Fuzzy Goals for Water Quality Management

The quantity of interest is the concentration level, $C_{il}$, of the water quality parameter and the fraction removal level (treatment level), $x_{imn}$, of the pollutant. The quantities, $x_{imn}$, are the fraction removal levels of the pollutant $n$ from the discharger $m$ to control the water quality parameter $i$. 
Recent Modelling Tools

Fuzzy Goals of the Pollution Control Agency  Fuzzy Goal \( E_{il} \): Make the concentration level, \( C_{il} \), of the water quality parameter \( i \) at the checkpoint \( l \) as close as possible to the desirable level, \( D_{il} \), so that the water quality at the checkpoint \( l \) is enhanced with respect to the water quality parameter \( i \), for all \( i \) and \( l \).

Fuzzy Goals of the Dischargers  Fuzzy Goal \( F_{imn} \): Make the fraction removal level \( x_{imn} \) as close as possible to the aspiration level \( L_{imn} x \) for all \( i \), \( m \), and \( n \).

The membership function corresponding to the decision \( Z \) is given by

\[
\mu_z(X) = \min_{i,m,n} \left( \mu_E(C_i), \mu_{F_{imn}}(x_{imn}) \right)
\]

where \( X \) is the space of alternatives composed of \( C_i \) and \( x_{imn} \). The corresponding optimal decision, \( X^* \), is given by

\[
X^* = \max_y \left[ \mu_z(X) \right]
\]

Membership Functions for the Fuzzy Goals  Goal \( E_{il} \): The membership function for the fuzzy goal \( E_{il} \) is constructed as follows. The desirable level, \( D_{il} \), for the water quality parameter \( i \) at checkpoint \( l \) is assigned a membership value of 1. The minimum permissible level, \( L_{il} \), is assigned a membership value of zero. The membership function for the fuzzy goal \( E_{il} \) is expressed as

\[
\mu_E(C_i) = \begin{cases} 
0 & \text{if } C_i \leq L_{il} \\
\frac{C_i - L_{il}}{D_{il} - L_{il}} & \text{if } L_{il} < C_i < C_{il} \\
1 & \text{if } C_i \geq C_{il}
\end{cases}
\]

With a similar argument, the membership function for the goal \( F_{imn} \) is written as

\[
\mu_{F_{imn}}(x_{imn}) = \begin{cases} 
0 & \text{if } x_{imn} \leq x_{imn}^L \\
\frac{x_{imn} - x_{imn}^L}{x_{imn}^M - x_{imn}^L} & \text{if } x_{imn}^L < x_{imn} \leq x_{imn}^M \\
1 & \text{if } x_{imn} \geq x_{imn}^M
\end{cases}
\]

These membership functions may be interpreted as the variation of satisfaction levels of the pollution control agency and the dischargers. The indices \( \alpha_E \) and \( \beta_{F_{imn}} \) determine the shape of the membership functions. \( \alpha_E = \beta_{F_{imn}} = 1 \) would result in linear membership functions.

Fuzzy Optimization Model  The optimization model is formulated to maximize the minimum satisfaction level, \( \lambda \), in the system. The model is expressed as
Applications

\[
\text{max } \lambda \quad (9.3.1)
\]
subject to
\[
\begin{align*}
\mu_{i}(C_{d}) & \geq \lambda \quad \forall \ i, \ l \quad (9.3.2) \\
\mu_{\text{max}}(x_{\text{min}}) & \geq \lambda \quad \forall \ i, \ m, \ n \quad (9.3.3) \\
C_{d}^{L} & \leq C_{d} \leq C_{d}^{U} \quad \forall \ i, \ l \quad (9.3.4) \\
x_{\text{min}}^{L} & \leq x_{\text{min}} \leq x_{\text{min}}^{U} \quad \forall \ i, \ m, \ n \quad (9.3.5) \\
x_{\text{max}}^{L} & \leq x_{\text{max}} \leq x_{\text{max}}^{U} \quad \forall \ i, \ m, \ n \quad (9.3.6) \\
0 & \leq \lambda \leq 1 \quad (9.3.7)
\end{align*}
\]

The crisp constraints (9.3.4) through (9.3.7) determine the space of alternatives. The constraints (9.3.4) are based on the water quality requirements set by the pollution control agency through the desirable and permissible limits of the water quality parameter \( i \). The aspiration level and maximum acceptable level of pollutant treatment efficiencies set by the dischargers are expressed in constraints (9.3.5) and (9.3.6). The constraints (9.3.2) and (9.3.3) define the parameter \( \lambda \) as the minimum satisfaction level in the system. The objective is to find \( X' \) corresponding to the maximum value \( \lambda' \) of the parameter \( \lambda \). The optimum value \( \lambda' \) corresponds to the maximized minimum (max–min) satisfaction level in the system. The solution \( X' \) is referred to as the best compromise solution to the multi-objective optimization problem.

The concentration level, \( C_{d} \), of the water quality parameter \( i \) at the checkpoint \( l \) can be related to the fraction removal level, \( x_{\text{min}} \), of the pollutant \( n \) from the discharger \( m \) to control the water quality parameter \( i \), though the transfer function that may be mathematically expressed as

\[
C_{d} = \sum_{m \in \text{dischargers}} \sum_{l \in \text{checkpoints}} f_{\text{min}}(L_{\text{dis}}, x_{\text{min}}) + \sum_{p \in \text{sources}} \sum_{l \in \text{checkpoints}} f_{\text{p}}(L_{\text{p}})
\]

where \( L_{\text{dis}} \) is the concentration of the pollutant \( n \) prior to treatment from the discharger \( m \) that affects the water quality parameter \( i \) at the checkpoint \( l \). \( L_{\text{p}} \) is the concentration of the pollutant \( n \) from the uncontrollable (nonpoint) source \( p \) (such as agricultural runoff, pollution due to sediments etc.) that affect the water quality parameter \( i \) at the checkpoint \( l \). The transfer functions \( f_{\text{min}}(\cdot, \cdot) \) and \( f_{\text{p}}(\cdot) \) represent the concentration levels of the water quality parameter \( i \) due to \( L_{\text{dis}}(1 - x_{\text{min}}) \) and \( L_{\text{p}} \), respectively. These transfer functions can be evaluated using appropriate mathematical models that determine the spatial and temporal distribution of the water quality parameter due to the pollutants in the river system. The solution of the optimization model (9.3.1) to (9.3.7) are \( X' \) and \( \lambda' \) where \( X' \) is the set of optimal fraction removal levels, and \( \lambda' \) is the maximized \( \lambda \). Thus,

\[
X' = \{ x_{\text{min}} \}
\]

where \( x_{\text{min}} \) is the fraction removal level of the pollutant \( n \) from the discharger \( m \) to control the water quality parameter \( i \).
An Example Application

Consider the hypothetical river network shown in Fig. 9.14 (Sasikumar and Mujumdar, 1992). The river network is discretized into 9 river reaches. Each reach receives a point-source of Biochemical Oxygen Demand (BOD) load from a discharger located at the beginning of the reach. The only pollutant in the system is the point source of BOD waste load. The water quality parameter of interest is the dissolved oxygen deficit (DO deficit) at 23 checkpoints due to the point-sources of BOD. The data pertaining to the river flows and effluent flows are given in Table 9.2. The transfer function that expresses the DO deficit at a checkpoint in terms of the concentration of point-source of BOD and the fraction removal levels can be obtained using the one-dimensional steady state Streeter-Phelps BOD-DO equations. Refer, for example, Thomann and Mueller, 1987 and James and Elliot, 1993.

Since only one pollutant and one water quality parameter are considered, the suffixes \( i \) and \( n \) are dropped from the constraints and objective function for convenience. Denoting the DO deficit at the water quality checkpoint \( l \) by \( C_l \) and the fraction removal level for the \( m \)th discharger by \( x_m \), and using linear membership functions for the fuzzy goals (i.e. \( \alpha_{l-1} = \beta_{l-1} = 1 \)), the Fuzzy LP formulation can be written as follows:

\[
\text{max } \lambda
\]

subject to

\[
\frac{C_l^H - C_l}{C_l^T - C_l} \geq \lambda \quad \forall \ l
\]

\[
\frac{x_m - x_m^0}{x_m^H - x_m^0} \leq \lambda \quad \forall \ m
\]

\[
C_l^H \leq C_l \leq C_l^T \quad \forall \ l
\]
Recent Modelling Tools

\[
\max \left\{ x^l_m, x^m_m \right\} \leq x^u_m \leq x^H_m \quad \forall m
\]

\[0 \leq \lambda \leq 1\]

where \( C^H_m \) is the maximum acceptable DO-deficit. Details of the membership functions for the fuzzy goals are given in Table 9.3. Two typical membership functions corresponding to the fuzzy goals \( E_3 \) (goal of the pollution control agency related to the DO deficit at checkpoint 3), and \( F_9 \) (goal related to the fraction removal level for discharger 9) are shown in Fig. 9.15 and Fig. 9.16 respectively. A minimal fraction removal level of 0.30 is imposed by the pollution control agency on all the dischargers (i.e., \( x^m_m \geq 0.3 \forall m \)).

### Table 9.3: Details of Membership Functions

<table>
<thead>
<tr>
<th>Checkpoints ( r )</th>
<th>In reach ( r )</th>
<th>( C^l_m ) (mg/L)</th>
<th>( C^u_m ) (mg/L)</th>
<th>Discharger</th>
<th>( x^l_m )</th>
<th>( x^u_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_1 )</td>
<td>1</td>
<td>3.5</td>
<td>0.0</td>
<td>( D_1 )</td>
<td>0.25</td>
<td>0.75</td>
</tr>
<tr>
<td>( r_2 )</td>
<td>2</td>
<td>3.0</td>
<td>0.5</td>
<td>( D_2 )</td>
<td>0.35</td>
<td>0.80</td>
</tr>
<tr>
<td>( r_3 )</td>
<td>3</td>
<td>3.5</td>
<td>0.0</td>
<td>( D_3 )</td>
<td>0.30</td>
<td>0.85</td>
</tr>
<tr>
<td>( r_4 )</td>
<td>4</td>
<td>3.0</td>
<td>0.0</td>
<td>( D_4 )</td>
<td>0.35</td>
<td>0.75</td>
</tr>
<tr>
<td>( r_5 )</td>
<td>5</td>
<td>3.0</td>
<td>0.5</td>
<td>( D_5 )</td>
<td>0.25</td>
<td>0.90</td>
</tr>
<tr>
<td>( r_6 )</td>
<td>6</td>
<td>4.0</td>
<td>1.0</td>
<td>( D_6 )</td>
<td>0.35</td>
<td>0.90</td>
</tr>
<tr>
<td>( r_7 )</td>
<td>7</td>
<td>3.5</td>
<td>1.5</td>
<td>( D_7 )</td>
<td>0.35</td>
<td>0.85</td>
</tr>
<tr>
<td>( r_8 )</td>
<td>8</td>
<td>4.0</td>
<td>1.5</td>
<td>( D_8 )</td>
<td>0.30</td>
<td>0.75</td>
</tr>
</tbody>
</table>

The results are summarized in Table 9.4.

The optimal value of the minimum satisfaction level, \( \lambda \), in the system is 0.2283. The upper and lower bounds of \( \lambda \) reflect two extreme scenarios in the system. The upper bound, \( \lambda = 1 \), indicates that all the goals have been completely satisfied and therefore represents a no-conflict scenario. The lower bound, \( \lambda = 0 \), indicates that at least one goal has a zero satisfaction level and therefore represents a conflict scenario. Any intermediate value of \( \lambda \) represents

![Fig. 9.15](image-url)  
**Fig. 9.15** | Membership Function for Goal \( E_3 \)
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![Graph](image)

**Fig. 9.16** Membership Function for Goal $F_G$

**Table 9.4** Results of Fuzzy Optimization

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<tr>
<th>Discharger</th>
<th>Fraction Removal Level</th>
<th>River Reach $R_e$</th>
<th>Minimum DO Concentration (mg/L)</th>
</tr>
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<tr>
<td>($D_1$)</td>
<td>0.64</td>
<td>$r_1$</td>
<td>9.89</td>
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<tr>
<td>($D_2$)</td>
<td>0.70</td>
<td>$r_2$</td>
<td>8.76</td>
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<tr>
<td>($D_3$)</td>
<td>0.72</td>
<td>$r_2$</td>
<td>8.50</td>
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<tr>
<td>($D_4$)</td>
<td>0.66</td>
<td>$r_2$</td>
<td>8.80</td>
</tr>
<tr>
<td>($D_5$)</td>
<td>0.70</td>
<td>$r_2$</td>
<td>9.17</td>
</tr>
<tr>
<td>($D_6$)</td>
<td>0.75</td>
<td>$r_2$</td>
<td>7.65</td>
</tr>
<tr>
<td>($D_7$)</td>
<td>0.77</td>
<td>$r_2$</td>
<td>6.90</td>
</tr>
<tr>
<td>($D_8$)</td>
<td>0.74</td>
<td>$r_2$</td>
<td>6.61</td>
</tr>
<tr>
<td>($D_9$)</td>
<td>0.49</td>
<td>$r_2$</td>
<td>6.07</td>
</tr>
</tbody>
</table>

the degree of conflict that exists in the system. The Fuzzy LP formulation aims at achieving a fair-compromise solution by reducing the degree of conflict in the system. A low value of $\mu$ indicates that a conflict scenario cannot be avoided in the system. The existence of a conflict scenario in water quality management problems is due to the compound effect of the conflicting objectives of the pollution control agency and the dischargers, and the relatively low assimilative capacity of the river network.

### 9.3.3 Fuzzy LP for Reservoir Operation

We illustrate here an application of a fuzzy LP model developed by Jairaj and Vedula (2003) to determine the annual relative yield of crops and its associated reliability, for a single reservoir irrigating multiple crops. Reservoir inflow, rainfall in the command area, and deep percolation from irrigated area are treated as random, while potential evapotranspiration is modelled as a fuzzy variable. The motivation for modelling potential evapotranspiration as a fuzzy variable is because of the uncertainty associated with determining various factors such as evaporation and crop factor that affect it. The probability distribution of inflow and rainfall in the command area are estimated from historical data.
Model Formulation

The objective is to maximize the annual crop water utilization in the reservoir command area contributing to the optimal growth of crops and leading to optimal production. The sum of weighted actual evapotranspiration (AET) of crops over all periods of a year is maximized. The yield response factor is used as the weighting factor.

\[
\text{Maximize } Z = \sum_{c} \sum_{t} k_{ct} AET_{ct}^c
\]

where \( k_{ct} \) is the yield response factor for the crop \( c \) in each period \( t \), which in turn depends on the growth stage of the crop to which period \( t \) belongs.

We need the maximum and the minimum values of \( Z \) to construct its membership function in the fuzzy model. These are determined using a deterministic model with estimated values of PET for each crop in each period.

The overall problem is solved in two phases; Phase 1: Chance Constrained Mixed Integer Linear Programming (CCMILP) with deterministic PET values, and Phase 2: Chance Constrained Mixed Integer Fuzzy Linear Programming (with PET of crops defined by fuzzy sets). Solution of Phase 1 gives estimates of the minimum and maximum values of \( Z \), corresponding to known minimum and maximum values of PET, respectively, which are used in the Phase 2 model.

Phase 1: CCMILP Model (Deterministic PET)

Reservoir Storage Continuity Constraints

The reservoir water balance is governed by the storage continuity as

\[
ST_{t+1} = ST_t + I_t - REL_t - EVP_t - SPILL_t \quad \forall \ t
\]

where

- \( ST_t \) = storage at the beginning of period \( t \),
- \( I_t \) = inflow in period \( t \),
- \( REL_t \) = reservoir release in period \( t \),
- \( EVP_t \) = evaporation loss in period \( t \),
- \( SPILL_t \) = spill in period \( t \).

Inflow and release are treated as random variables, and \( SPILL_t \) is considered deterministic.

The storage continuity equation with evaporation approximation and a term for spill [following Section (5.1.2)] is written as

\[
(1 + a_t)ST_{t+1} = (1 - a_t)ST_t + I_t - REL_t - SPILL_t - A_0 \epsilon_t \quad \forall \ t
\]

where \( A_0 \) is an arbitrarily large constant, and \( \epsilon_t \) is a \((0, 1)\) integer variable given by

\[
\xi_t \leq ST_{t+1}/ST_{max} \quad \forall \ t
\]
When spill occurs, $ST_{t+1}$ equals $ST_{\text{max}}$ and $t_0$ equals 1; otherwise $ST_{t+1}$ is less than $ST_{\text{max}}$ and $t_0$ equals zero.

**Soil Moisture Balance** Following the notation in Section 7.3, the soil moisture balance equation is written as

$$SM_t^* = SM_{t-1}^* + RAIN_t + FC^* (D_{t-1}^* - D_t^*) - PET_t^* - D_P^* = SM_{t+1}^* - D_t^*$$

Soil moisture is bounded by its value at the wilting point ($WP^*$) and at field capacity ($FC^*$), $WP^* \leq SM_t^* \leq FC^*$.

**Actual Evapotranspiration** Variation of $AET_t$ is incorporated as shown in Fig. 9.17. The figure shows the variation of $AET_t/PET$ with the available soil moisture (soil moisture above the wilting point) over the root depth. Up to a fractional level $f$ of the available soil moisture from that at field capacity, the plant will be able to draw water to its full requirement, thereby making $AET_t$ equal to $PET_t$. Below this level, $AET_t$ will be less than $PET_t$ and the ratio, $AET_t/PET$, decreases as in Fig. 9.17, resulting in water stress.

That is,

$$AET_t = \begin{cases} 
PET_t & \text{if } SM_t \geq (1-f)SM_{\text{max}} \\
\frac{SM_t}{(1-f)SM_{\text{max}}}PET_t & \text{otherwise}
\end{cases}$$

where $SM_{\text{max}}$ is the maximum available soil moisture at field capacity (in depth units per unit of root depth), $SM_t$ is the available soil moisture at time $t$ (in depth units per unit of root depth), $f$ is the fraction of maximum available soil moisture, below that (soil moisture) at field capacity, up to which $AET_t = PET_t$, i.e. $AET < PET$ when the available soil moisture in the root zone is between zero (at wilting point) and $(1-f)SM_{\text{max}}$, and $AET = PET$ when the available soil moisture is between $(1-f)SM_{\text{max}}$ and $SM_{\text{max}}$ (at field capacity).

The following constraints are incorporated to define the nonlinear (piecewise linear) variation of $AET_t/PET_t$ with $SM_t$ in the model. Additional variables, $S_t^*$ and $M_t^*$, corresponding to the abscissa in the sloping and horizontal segments, respectively, in Fig. 9.17, are introduced.
Recent Modelling Tools

\[ \frac{\Delta ET_t^c}{\Delta T_t^c} = \frac{S_t^c}{(1 - f)SM_{\text{max}}^t} \quad \forall c, t \]

where

\[ SM_t^c = S_t^c + M_t^c \quad \forall c, t \]

\[ S_t^c \leq (1 - f) SM_{\text{max}}^t \quad \forall c, t \]

\[ M_t^c \leq f SM_{\text{max}}^t \quad \forall c, t \]

We should ensure that \( M_t^c \) comes into play only after \( S_t^c \) reaches its maximum value. This can be done by using a technique similar to the one used to prevent spill before the reservoir is full. Thus in order to prevent the variable \( M_t^c \) taking on a positive value before \( S_t^c \) reaches \((1 - f) SM_{\text{max}}^t\) (equal to the maximum value of \( S_t^c \)), the following two constraints are imposed.

\[ M_t^c \leq LM \zeta_t^c \quad \forall c, t \]

where \( LM \) is an arbitrary large constant, and \( \zeta_t^c \) is a \((0, 1)\) integer variable given by

\[ \zeta_t^c = \frac{S_t^c}{(1 - f)SM_{\text{max}}^t} \quad \forall c, t \]

This ensures the choice of appropriate values of \( S_t^c \) and \( M_t^c \) in the correct sequence for the given \( SM_t^c \).

**Deep Percolation**

This water is not available for plant growth. To specify the condition that deep percolation occurs in a period only when the soil moisture at the end of the period reaches field capacity, the following constraints are imposed.

\[ DP_t^c \leq B \beta_t^c \quad \forall c, t \]  \( (9.3.1) \)

where \( DP_t^c \) is the deep percolation for the crop \( c \) during period \( t \), \( B \) is an arbitrarily large constant, and \( \beta_t^c \) is a \((0, 1)\) integer variable, given by

\[ \beta_t^c = \frac{SM_{t+1}^c}{SM_{\text{max}}^t} \quad \forall c, t \]

When \( SM_{t+1}^c = SM_{\text{max}}^t \), \( \beta_t^c = 1 \), and therefore \( DP_t^c \) is nonzero. On the other hand, when \( SM_{t+1}^c < SM_{\text{max}}^t \), \( \beta_t^c = 0 \), and therefore \( DP_t^c = 0 \).

**Chance Constraints**

Two sets of chance constraints, one associated with release from the reservoir (random variable), and the other associated with deep percolation (random variable) in the irrigated command area, are used.

**Reliability Constraint**

Chance constraint defines the release policy from the reservoir in any period. The policy states that the probability of release in any time period equalling or exceeding the total irrigation allocation to all crops in that period is greater than or equal to a specified value, termed as the reliability level (Section 6.3).
where $Pr$ denotes probability, $REL_t$ is the release from the reservoir during period $t$ (volume units), $\eta$ is the ratio of the sum total of crop water allocations at the field level to the release at the reservoir (efficiency factor), $\chi_c^t$ is the irrigation allocation to crop $c$ in time period $t$ (depth units), $A_c$ is the area under crop $c$ (area units), and $\alpha_t$ is the reliability level specified for period $t$.

Substituting for $REL_t$ from the chance constraint in the storage continuity equation

$$Pr \left[ REL_t \geq \frac{1}{\eta} \sum_c (A_c \chi_c^t) \right] \geq \alpha_t \quad (9.3.2)$$

Using the linear decision rule, $REL_t = ST_t + I_t - EVP_t - b_t - SPILL_t$, where $b_t$ is a deterministic parameter (Section 6.2.1), the deterministic equivalent is obtained as

$$Pr \left[ (1 + \alpha_t)ST_{t-1} - (1 - \alpha_t)ST_t + A_0 \beta_t + SPILL_t + \frac{1}{\eta} \sum_c (A_c \chi_c^t) \leq I_t \right] \geq \alpha_t \quad \forall \ t$$

Deep percolation, being a random variable, the constraint, Eq (9.3.1), for $DP_t^c$ is written as a chance constraint

$$Pr \left[ (1 + \alpha_t)ST_{t+1} - (1 - \alpha_t)ST_t + A_0 \beta_t + SPILL_t + \frac{1}{\eta} \sum_c (A_c \chi_c^t) \leq I_t \right] \geq \alpha_t \quad \forall \ t$$

The chance constraints are modified in the light of the soil moisture balance equation, as

$$Pr \left[ SM_{t+1}^c - SM^c_t - AET_t^c - \frac{1}{\eta} \sum_c (D_t^c - D_{t+1}^c) \geq \beta_t \right] \geq B_0 \quad \forall \ c, t$$

and

$$Pr \left[ SM_{t+1}^c - SM^c_t - AET_t^c - \frac{1}{\eta} \sum_c (D_t^c - D_{t+1}^c) \leq \beta_t \right] \geq B_0 \quad \forall \ c, t$$

The deterministic equivalents of these chance constraints being

$$SM_{t+1}^c - SM^c_t - AET_t^c - \frac{1}{\eta} \sum_c (D_t^c - D_{t+1}^c) \leq \beta_t \quad \forall \ c, t$$

where $RAIN_t^c$ is the rainfall in period $t$ with exceedance probability $\beta_t$.
The reservoir and canal capacity constraints for each period are specified appropriately.

**Canal Capacity Constraint** A canal capacity constraint is used to specify that the capacity of the canal carrying water from the reservoir to the field is enough to carry water needed for field irrigation requirements. Thus,

$$\frac{1}{\eta} \left( \sum_{c} A_c x_c^t \right) \leq CAP \quad \forall t$$

where CAP is the canal capacity.

**Model** The objective function along with the associated constraints for Phase 1 model are:

Maximize $Z = \sum_{t} k^t_i \cdot AET^t_i$

subject to

$$SPILL_t \leq B \xi_t \quad \forall t$$

$$\xi_t \leq ST_{i,t}/ST_{\text{max}} \quad \forall t$$

$$SM^t_i \leq SM_{\text{max}}^t \quad \forall c, t$$

$$AET^t_i = \frac{S_i^t}{PET_i} \quad \forall c, t$$

$$SM^t_i = S_i^t + M_i^t \quad \forall c, t$$

$$S_i^t \leq (1 - f) SM_{\text{max}}^t \quad \forall c, t$$

$$M_i^t \leq f SM_{\text{max}}^t \quad \forall c, t$$

$$M_i^t \leq LM \xi_t \quad \forall c, t$$

$$\xi_t = \frac{S_i^t}{(1 - f) SM_{\text{max}}^t} \quad \forall c, t$$

$$\beta_i^t \leq \frac{SM_{\text{max}}^t}{SM_{\text{max}}^t} \quad \forall c, t$$

$$(1 + \alpha_i) \beta_i^t - (1 - \alpha_i) \beta_i^t + B \xi_t + A_i \xi_t + \frac{1}{\eta} \sum_{c} (x_c^t \leq I^c_{\text{max}}) \quad \forall t$$

$$SM_{\text{max}}^t D_{i+1}^t + B \beta_i^t \cdot AET^t_i - x_i^t - SM_i^t D_i^t - SM_{\text{max}}^t (D_{i+1}^t - D_i^t) \geq RAIN_{i+1}^t \quad \forall c, t$$

$$SM_{\text{max}}^t D_{i+1}^t + AET_i^t - x_i^t - SM_i^t D_i^t - SM_{\text{max}}^t (D_{i+1}^t - D_i^t) \leq RAIN_i^t \quad \forall c, t$$
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The model is to be solved as a mixed integer linear programming problem.

**Phase 2: Fuzzy Model (Varying PET)**

*Membership function of PET.* Based on the minimum and maximum values of \( \text{PET}_t \) (as computed from data on evaporation and crop factors) denoted by \( \text{PET}_{t}^L \) and \( \text{PET}_{t}^U \) respectively, the membership function of \( \text{PET}_t \) is constructed as in Fig. 9.18.

\[
\frac{1}{p_t} \left( \sum_{j} A_{tj} x_j^c \right) \leq \text{CAP} \quad \forall t
\]

\[
b_{t} \leq \text{ST}_{\text{max}} \quad \forall t
\]

\[
b_{t}, \text{SPILL}_{c}, SM_{t}^c, S_{c}^t, M_{c}^t, AET_{c}^t, x_j^c \geq 0 \quad \forall c, t
\]

\[
\beta_j^c, \gamma_{c}^t, \zeta_{c}^t \text{ integer} \leq 1 \quad \forall c, t
\]

The model is to be solved as a mixed integer linear programming problem.

**Membership Function for the Objective Function, Z** Let \( Z_{L} \) and \( Z_{U} \) be the lower and upper bounds for \( Z \) as determined in the Phase 1 model, with appropriate values of \( \text{PET}_t \) in each case (\( Z_{L} \) is obtained by using the minimum values of \( \text{PET}_t \) for each \( t \), and \( Z_{U} \) by using the maximum values of \( \text{PET}_t \) for each \( t \)). Potential evapotranspiration itself is estimated from the reference evaporation and crop factor.

The membership function of the objective \( \mu(Z) \), is defined as in Fig. 9.19, assuming a linear variation.

**Model** Based on the membership function for \( \text{PET}_t^c \), the fuzzy constraint containing \( \text{PET}_t^c \) is transformed as

\[
(1 - f) \cdot SM_{\text{max}} \cdot AET_{t}^c - (p_t^c + d_t^c) S_t^c + \lambda d_t^c S_t^c = 0 \quad \forall c, t
\]

where \( \lambda \) is the membership function value to be determined.
Note that this equation is nonlinear in nature due to the product of the two variables $\lambda$ and $S'_c$. A linearized approximation (similar to the approximation used for hydropower optimization problem, Section 7.5) is used:

$$\lambda S'_c = A_0 S'_c + \lambda S'(s) - A_0 S'_0 \quad \forall c, t$$

where $\lambda$ is the membership function value to be computed, $A_0$ is a trial value (assumed in a given run) of the membership function, $\lambda S'_c$ is a trial value (assumed in a given run) of $S'_c$ for crop $c$ during period $t$.

Substituting for $\lambda S'_c$ from the above equation yields

$$(1 - f) SM_{max} AET' - (f' + d') - A_0 S'_c d'_t \lambda = A_0 S'_c d'_t \quad \forall c, t$$

The objective function is transformed based on the membership function of $Z$, defined in Fig. 9.19 as

$$\begin{align*}
\text{Maximize } & \lambda (Z_U - Z_L) - \sum_{c,t} k_i AET' S' - Z_L \\
\text{subject to } & 0 \leq \lambda \leq 1 \quad \text{along with the other pertinent constraints as discussed earlier.}
\end{align*}$$

The model is solved repeatedly by assigning trial initial values for $A_0$ and $S'_c$ for each crop $c$ in each period $t$. The value of $\lambda$ and $S'_c$ obtained from the model solution are assigned as the initial values $A_0$ and $S'_c$ in the next run, and the procedure is repeated successively, till convergence is reached i.e. $\lambda = A_0$ and $S'_c = S'_c$, within specified tolerance limits. The procedure converges in a few iterations in application.

**Annual Relative Crop Yield (ARY)**

With the solution of Phase 2, the optimal values of $AET$ and $PET$ are known for each time period for each crop. Then the resulting relative yield of each crop can be computed and the sum of the relative yields of all crops obtained. This sum, $ARY$, is considered as an integral measure of the crop yield for the entire irrigated area. This is computed based on the expression for the sum of relative yields of multiple crops (Vedula and Nagesh Kumar, 1996).
where, $c$ is the index of crop, $k'_{yg}$ is the yield response factor for the growth stage $g$ of the crop $c$ (assumed to be same in each of the periods $t$ within the growth stage $g$), $NGS$ is the number of growth stages within the growing season of a crop, and $NC$ is the number of crops.

The tradeoff between the reliability and $ARY$ is obtained as shown in Fig. 9.20, for an existing reservoir in Karnataka.

![Fig. 9.20 Annual Relative Yield (ARY)](image)

It can be seen that for given probability of rainfall, the annual relative yield decreases at higher levels of specified reliability, as should be expected.

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