Introduction and Objectives

Simplex method is the most popular method used for the solution of Linear Programming Problems (LPP).

Objectives
- To discuss the motivation of simplex method
- To discuss Simplex algorithm
- To demonstrate the construction of simplex tableau

Motivation of Simplex method
- Solution of a LPP, if exists, lies at one of the vertices of the feasible region.
- All the basic solutions can be investigated one-by-one to pick up the optimal solution.
- For 10 equations with 15 variables there exists $^{15}C_{10} = 3003$ basic feasible solutions!
- Too large number to investigate one-by-one.
- This can be overcome by simplex method

Simplex Method: Concept in 3D case
- In 3D, a feasible region (i.e., volume) is bounded by several surfaces
- Each vertex (a basic feasible solution) of this volume is connected to the other adjacent vertices by a straight line to each, being intersection of two surfaces.
- Simplex algorithm helps to move from one vertex to another adjacent vertex which is closest to the optimal solution among all other adjacent vertices.
- Thus, it follows the shortest route to reach the optimal solution from the starting point.

Thus there is no need to investigate all the basic feasible solutions. Sequence of basic feasible solutions on the shortest route is generated by simplex algorithm

General procedure of simplex method
Simplex method involves following steps
1. General form of given LPP is transformed to its canonical form (refer Lecture notes 1).
2. Find a basic feasible solution of the LPP (there should exist at least one).
3. Move to an adjacent basic feasible solution which is closest to the optimal solution among all other adjacent vertices.
4. Repeat until optimum solution is achieved.
   Step three involves ‘Simplex Algorithm’
Simplex Method

Let us consider the following LPP

Maximize \[ Z = 4x_1 - x_2 + 2x_3 \]
subject to \[ 2x_1 + x_2 + 2x_3 \leq 6 \]
\[ x_1 - 4x_1 + 2x_3 \leq 0 \]
\[ 5x_1 - 2x_2 - 2x_3 \leq 4 \]
\[ x_1, x_2, x_3 \geq 0 \]

Other subscript notations are self explanatory

10 is minimum.

Is there any possibility of further improvement?

Simplex Algorithm ... contd.

Set of equations, including the objective function is transformed to canonical form

\[ -4x_1 + x_2 - 2x_3 + 0x_1 + 0x_2 + 0x_3 + Z = 0 \]
\[ 2x_1 + x_2 + 2x_3 + 0x_1 + 0x_2 + 0x_3 = 6 \]
\[ x_1 - 4x_1 + 2x_3 + 0x_1 + 0x_2 + 0x_3 = 0 \]
\[ 5x_1 - 2x_2 - 2x_3 + 0x_1 + 0x_2 + 0x_3 = 4 \]

Basic feasible solution of above canonical form is \( x_1 = 6, x_2 = 0, x_3 = 4, x_1 = x_2 = x_3 = 0 \) and \( Z = 0 \)

\( x_1, x_2, x_3 \): Basic Variables; \( x_4, x_5, x_6 \): Nonbasic Variables

Simplex Algorithm ... contd.

Let us consider the following LPP

Maximize \[ Z = -4x_1 + x_2 + 2x_3 \]
subject to \[ 2x_1 + x_2 + 2x_3 = 6 \]
\[ x_1 - 4x_1 + x_3 = 0 \]
\[ 5x_1 - 2x_2 - 2x_3 = 4 \]
\[ x_1, x_2, x_3 \geq 0 \]

Note that \( x_4, x_5, x_6 \) are slack variables

Simplex Algorithm ... contd.

Symbolized form (for ease of discussion)

\[ (Z) \] \[ c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 + c_6x_6 + Z = b \]
\[ (x_1) \] \[ c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 + c_6x_6 = b_i \]
\[ (x_2) \] \[ c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 + c_6x_6 = b_i \]
\[ (x_3) \] \[ c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 + c_6x_6 = b_i \]

• The left-most column is known as basis as this is consisting of basic variables
• The coefficients in the first row \( c_1, \ldots, c_6 \) are known as cost coefficients.
• Other subscript notations are self explanatory

Simplex Algorithm ... contd.

This completes first step of algorithm. After completing each step (iteration) of algorithm, following three points are to be examined:

1. Is there any possibility of further improvement?
2. Which nonbasic variable is to be entered into the basis?
3. Which basic variable is to be exited from the basis?

Simplex Algorithm ... contd.

LPP is transformed to its standard form

Maximize \[ Z = -4x_1 + x_2 - 2x_3 + Z = 0 \]
subject to \[ 2x_1 + x_2 + 2x_3 = 6 \]
\[ x_1 - 4x_1 + 2x_3 = 0 \]
\[ 5x_1 - 2x_2 - 2x_3 = 4 \]
\[ x_1, x_2, x_3 \geq 0 \]

Simplex Algorithm ... contd.

Simplex Algorithm ... contd.

Is there any possibility of further improvement?

If any one of the cost coefficients is negative further improvement is possible.

Which nonbasic variable is to be entered?

Entering variable is decided such that the unit change of this variable should have maximum effect on the objective function. Thus the variable having the coefficient which is minimum among all cost coefficients is to be entered, i.e., \( x_4 \) is to be entered if cost coefficient \( c_4 \) is minimum.
**Simplex Algorithm ... contd.**

**Which basic variable is to be exited?**

After deciding the entering variable \( x_r \), \( x_j \) (from the set of basic variables) is decided to be the exiting variable if \( \frac{b_j}{c_{ij}} \) is minimum for all possible \( r \), provided \( c_{ij} \) is positive. \( c_{ij} \) is considered as pivotal element to obtain the next canonical form.

**Simplex Algorithm ... contd.**

**Pivotal operation as a refresher**

- Pivotal row is transformed by dividing it with the pivotal element. In this case, pivotal element is 1.
- For other rows: Let the coefficient of the element in the pivotal column of a particular row be \( \frac{a_{ij}}{a_{kk}} \). Let the pivotal element be \( \frac{a_{ij}}{a_{jj}} \). Then the pivotal row is multiplied by \( \frac{a_{kk}}{a_{jj}} \) and then subtracted from that row to be transformed. This operation ensures that the coefficients of the element in the pivotal column of that row becomes zero, e.g., \( \frac{Z}{x_i} = -4 \), \( m = 1 \). So, pivotal row is multiplied by \( \frac{a_{kk}}{a_{jj}} \) and then subtracted from that row to be transformed. This is subtracted from \( Z \) row, obtaining \( \frac{Z}{x_i} = -4 \), \( m = 1 \). So, pivotal row is multiplied by \( \frac{a_{kk}}{a_{jj}} \) and then subtracted from that row to be transformed.

**Simplex Algorithm ... contd.**

The canonical form obtained after third iteration

\[
\begin{align*}
\mathbf{Z} & = 4x_1 + x_2 + 2x_3 + 4x_4 + 0x_6 + Z = 10 \\
x_1 & = 0x_1 + 9x_2 - 2x_3 + 1x_4 + 2x_5 + 0x_6 + Z = 6 \\
x_2 & = 1x_1 - 4x_2 + 2x_3 + 0x_4 + 1x_5 + 0x_6 + Z = 0 \\
x_3 & = 0x_1 + 18x_2 - 12x_3 + 0x_4 + 5x_5 + 1x_6 + 4
\end{align*}
\]

The basic solution of above canonical form is \( x_1 = 0, x_2 = 6, x_3 = 4, x_2 = x_3 = 0 \) and \( Z = 0 \).

Note that cost coefficient \( c_j \) is negative. Thus optimum solution is not yet achieved. Further improvement is possible.

Thus another canonical form is obtained.

**Simplex Algorithm ... contd.**

After the pivotal operation, the canonical form obtained as follows

\[
\begin{align*}
\mathbf{Z} & = 0x_1 - 15x_2 + 6x_3 + 4x_4 + 0x_6 + Z = 0 \\
x_1 & = 0x_1 + 9x_2 - 2x_3 + 1x_4 + 2x_5 + 0x_6 + Z = 6 \\
x_2 & = 1x_1 - 4x_2 + 2x_3 + 0x_4 + 1x_5 + 0x_6 + Z = 0 \\
x_3 & = 0x_1 + 18x_2 - 12x_3 + 0x_4 + 5x_5 + 1x_6 + 4
\end{align*}
\]

The basic solution of above canonical form is \( x_1 = 0, x_2 = 6, x_3 = 4, x_2 = x_3 = 0 \) and \( Z = 0 \).

Note that cost coefficient \( c_j \) is negative. Thus optimum solution is not yet achieved. Further improvement is possible.
Simplex Algorithm … contd.

Note that cost coefficient $c_4$ is negative. Thus optimum solution is not yet achieved. Further improvement is possible.

Entering variable (Following the similar procedure)

$x_4$ is the entering variable. Thus $c_4 (= 4)$ is the pivotal element and $x_4$ is to be replaced by $x_3$ in the basis.

Thus another canonical form is obtained.

Exiting variable (Following the similar procedure)

$x_3$ is the exiting variable for the next step of calculation.

Note that cost coefficient $c_3$ is negative. Thus optimum solution is not yet achieved. Further improvement is possible.

The canonical form obtained after fourth iteration

\[
\begin{align*}
(Z) & \quad 0x_1 + 0x_2 + 0x_3 + 1x_4 + \frac{1}{4} x_5 + \frac{1}{2} x_6 + Z = \frac{22}{3} \\
(x_1) & \quad 0x_1 + 0x_2 + 1x_3 + \frac{1}{4} x_4 + \frac{1}{8} x_5 - \frac{1}{2} x_6 = 1 \\
(x_2) & \quad 1x_1 + 0x_2 + 0x_3 + \frac{1}{4} x_4 - \frac{1}{36} x_5 + \frac{5}{36} x_6 = \frac{14}{9} \\
(x_3) & \quad 0x_1 + 1x_2 + 0x_3 + \frac{1}{6} x_4 - \frac{7}{36} x_5 - \frac{1}{36} x_6 = \frac{8}{9}
\end{align*}
\]

The basic solution of above canonical form is $x_1 = 14/9, x_3 = 1, x_4 = x_5 = 0$ and $Z = 22/3$.

Simplex Algorithm … contd.

Construction of Simplex Tableau: General notes

Calculations shown till now can be presented in a tabular form, known as simplex tableau.

After preparing the canonical form of the given LPP, first simplex tableau is constructed.

After completing each simplex tableau (iteration), few steps (somewhat mechanical and easy to remember) are followed. Logically, these steps are exactly similar to the procedure described earlier.

Construction of Simplex Tableau: Basic steps

Check for optimum solution:

1. Investigate whether all the elements (coefficients of the variables headed by that column) in the first row (i.e., Z row) are nonnegative or not. If all such coefficients are nonnegative, optimum solution is obtained and no need of further iterations.

   If any element in this row is negative follow next steps to obtain the simplex tableau for next iteration.

Operations to obtain next simplex tableau:

2. Identify the entering variable (described earlier) and mark that column as Pivotal Column.

3. Identify the exiting variable from the basis as described earlier and mark that row as Pivotal Row.

4. Mark the coefficient at the intersection of Pivotal Row and Pivotal Column as Pivotal Element.
Construction of Simplex Tableau: Basic steps

Operations to obtain simplex tableau...contd.:
5. In the basis, replace the exiting variable by entering variable.
6. Divide all the elements in the pivotal row by pivotal element.
7. For any other row, identify the elementary operation such that the coefficient in the pivotal column, in that row, becomes zero. Apply the same operation for all other elements in that row and change the coefficients.
Follow similar procedure for all other rows.

Consider the same problem discussed before.
Canonical form of this LPP is

\[-4x_1 + x_2 - 2x_3 + 0x_4 + 0x_5 + 8x_6 + Z = 0\]
\[2x_1 + x_2 + 2x_3 + 1x_4 + 0x_5 + 0x_6 = 6\]
\[x_1 - 4x_2 + 2x_4 + 0x_3 + 1x_5 + 0x_6 = 0\]
\[5x_1 - 2x_2 - 2x_3 + 0x_4 + 0x_5 + 1x_6 = 4\]

Corresponding simplex tableau

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basis</th>
<th>Variables</th>
<th>( Z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( s_1 )</td>
<td>1 -4 1 -2 0 0 0 0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>( s_2 )</td>
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**Construction of Simplex Tableau: example**

Successive iterations

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<th>Variables</th>
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</tr>
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<td>4</td>
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</table>

**Final Tableau**

<table>
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<th>Z</th>
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Optimum solution is

\[ Z = \frac{22}{3} = 7.333 \]

\[ x_1 = \frac{14}{3} = 4.667 \]

\[ x_2 = \frac{9}{2} = 4.5 \]

\[ x_3 = -1 \]

**Construction of Simplex Tableau: A note**

It can be noted that at any iteration the following two points must be satisfied:

1. All the basic variables (other than Z) have a coefficient of zero in the Z row.
2. Coefficients of basic variables in other rows constitute a unit matrix.

Violation of any of these points at any iteration indicates a wrong calculation. However, reverse is not true.

**Final results from Simplex Tableau**

All the elements in the first row (i.e., Z row), at iteration 4, are nonnegative. Thus, optimum solution is achieved.

Optimum solution is

\[ Z = \frac{22}{3} = 7.333 \]

\[ x_1 = \frac{14}{3} = 4.667 \]

\[ x_2 = \frac{9}{2} = 4.5 \]

\[ x_3 = 1 \]

**Thank You**